Homework 4.1 (bc) **QUESTIONS**

$$A = E^{-1} E A = E^{-1} \begin{bmatrix} I_{3\times3} & B \\ 0 & 0 \end{bmatrix}$$

$N^T A_1 = 0$ $\quad \begin{bmatrix} A_1 & N \end{bmatrix} \begin{bmatrix} I_{3\times3} & B \\ 0 & 0 \end{bmatrix}$

$$A = A_1 \begin{bmatrix} I_{3\times3} & B \end{bmatrix} + N \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$$

$N^T A = 0$ $\qquad \boxed{N(A^T) = ?}$?

skip

$\underline{R(A)}$ $\quad \boxed{N(A^T)}$ $\qquad \underline{R(A^T)}$, $\underline{N(A)}$

?

$A = A_1 \begin{bmatrix} I_{3\times3} & B \end{bmatrix}$

A_1 basis for $R(A)$

$A = E^{-1} \begin{bmatrix} I_{3\times3} & B \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} I_{3\times3} \\ B^T \end{bmatrix}$

Basis $R(A^T)$

$A =$ [E] [ ] = [ ]

$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} B \\ -I \end{bmatrix} = 0$

0 $\quad R(A^T$

basis for $R(A^T)$ is cols of $\begin{bmatrix} I \\ B^T \end{bmatrix}$

basis for $N(A)$ is cols of $\begin{bmatrix} B \\ -I \end{bmatrix}$

$E^{-1} = [A_1 \; N]$ $\quad N \neq E_2^T$

$\rightarrow N = \begin{bmatrix} B \\ -I \end{bmatrix}$

$E^{-1}E$

$AN = E^{-1}\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$EE^{-1} = I$

$[E][A_1 \; N]$ ★

E_2 → basis of $N(A^T)$

$EE^{-1} = I$

$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}[A \; X] = \begin{bmatrix} E_1A_1 & E_1X \\ E_2A_1 & E_2X \end{bmatrix}$

$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$E_2A_1 = 0$

$N(A^T)$ basis is cols of E_2^T

$E_2A = E_2A_1[I_{3\times3} \; B]$, where $E_2A_1 = 0$

Orthogonal = perpendicular $x^T y = 0$

a set of vectors is or<u>thono</u>r<u>mal</u> $[v_1 \cdots v_n]$

(*)
- $v_i^T v_j = 0 \quad i \neq j$ ea. vector is $\perp$ to ea. other
- $v_i^T v_i = 1$ ea. vector has length 1.

often use orthonormal coord systems because they are easy to invert.

$$x = V x' \qquad V = [v_1 \cdots v_n]$$

($v_1 \cdots v_n$: orthonormal)

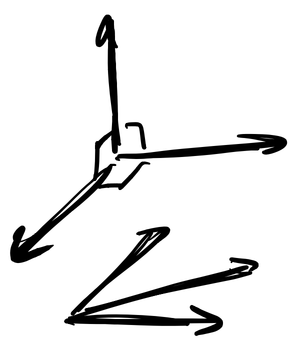
$$V^{-1} x = x'$$

$$\boxed{V^{-1} = V^T}$$

$$V^T V = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

$$= \begin{bmatrix} v_1^T v_1 & \cdots & v_1^T v_n \\ v_n^T v_1 & \cdots & v_n^T v_n \end{bmatrix}$$

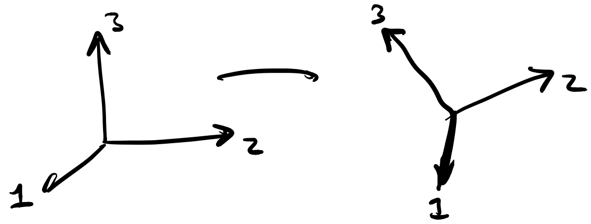
$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I$$



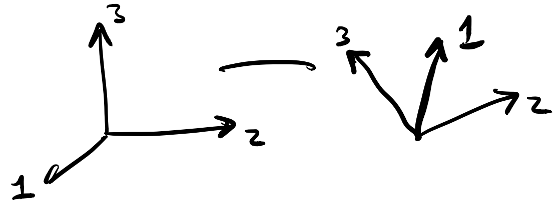
← inverting hard

Orthonormal Transformations $V = [v_1 \cdots v_n]$
- Rotation matrices ($\det(V) = 1$) ←
- Reflection matrices ($\det(V) = -1$)

Rotation Matrix:



Rotation & Reflection



More on Evals & Evecs:

Diagonalization of $A \in \mathbb{R}^{n \times n}$

$$A = PDP^{-1}$$

$$= \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

right eigenvectors — eigen values — rows are left eigenvectors

$w_i^T v_j = 0 \quad \forall i \neq j \Leftarrow P^{-1} P = I$

(P^{-1}: rows are left evecs; P: cols are right evecs)

Length of eigenvectors doesn't matter...

should be eigensubspace as opposed to eigenvector if you change the length of the eigen vectors you find how does the diagonalization change?

Constructing A from $\underline{P}$ (right evecs) and $\underline{D}$ (evals)

$\rightarrow A = PDP^{-1}$

what if we choose P with different length cols

$$P' = [v_1 e_1 \cdots v_n e_n] = \underbrace{[v_1 \cdots v_n]}_{P} \underbrace{\begin{bmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{bmatrix}}_{E} = PE$$

$$P' D (P')^{-1} \qquad (P')^{-1} = (PE)^{-1} = E^{-1}P^{-1}$$

$$\underline{P} E D E^{-1} \underline{P}^{-1}$$

$$P \begin{bmatrix} e_1 & & 0 \\ & \ddots & \\ 0 & & e_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} e_1^{-1} & & 0 \\ & \ddots & \\ 0 & & e_n^{-1} \end{bmatrix} P^{-1}$$

$$P \begin{bmatrix} e_1 \lambda_1 e_1^{-1} & & 0 \\ & \ddots & \\ 0 & & e_n \lambda_n e_n^{-1} \end{bmatrix} P^{-1}$$

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

$$A = P D P^{-1}$$

- P → scale up length of cols of P
- D → eigen values don't change
- P^{-1} → scales down length of rows of P^{-1}

Repeated Eigenvalues:

Question: eigenvalues and eigenvectors of λI ?

all evals are λ

$\lambda I \underline{x} = \lambda \underline{x}$

any vector is an eigenvector

$$A = \begin{bmatrix} v_1 \, v_2 \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 I_{2\times 2} & \\ & \lambda_3 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} - w_1^T - \\ - w_2^T - \\ \vdots \end{bmatrix}$$

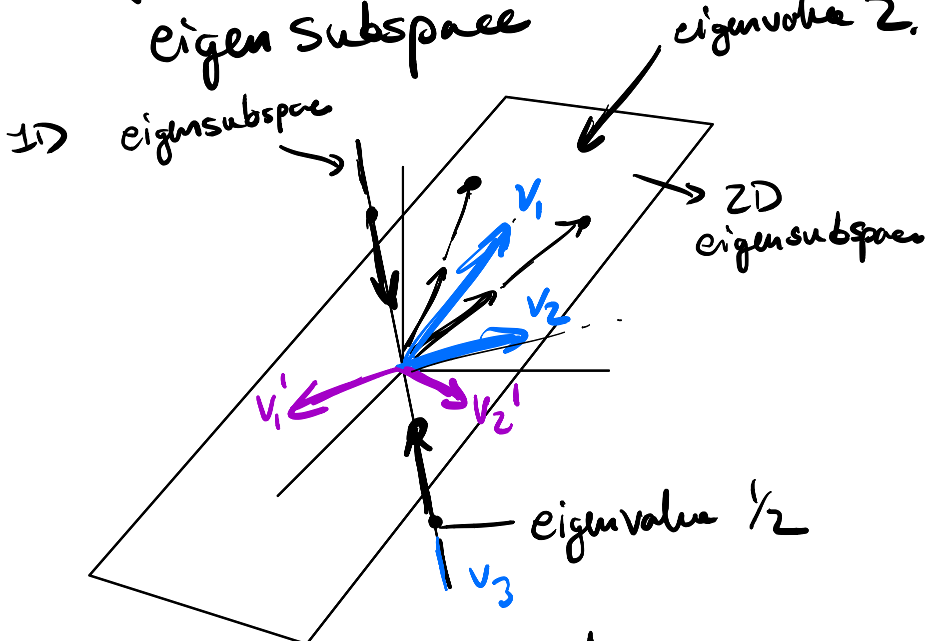
2D eigen subspace

$$\begin{bmatrix} v_1 \, v_2 \cdots \end{bmatrix} \begin{bmatrix} \lambda_1 Q Q^{-1} & \\ & \lambda_3 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} - w_1^T - \\ - w_2^T - \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} v_1 \, v_2 \cdots \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda_1 I & \\ & \lambda_3 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} [v_1 \, v_2] Q \cdots \end{bmatrix}}_{P'} \begin{bmatrix} \lambda_1 I & \\ & \lambda_3 & \\ & & \ddots \end{bmatrix} \underbrace{\begin{bmatrix} Q^{-1} \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix} \\ \vdots \end{bmatrix}}_{(P')^{-1}}$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} Q$$ = cols are a different basis for the 2D ← eigen subspace



$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} v_1' & v_2' & v_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1' & v_2' & v_3 \end{bmatrix}^{-1} \qquad \begin{bmatrix} v_1' & v_2' \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} Q$$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} Q & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\;0 & 1 \end{bmatrix} \begin{bmatrix} 2I & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\;0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} Q & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\;0 & 1 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2I & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 2QQ^{-1} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{-1}$$

Rearranging eigenvalues in diagonalization

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{n} v_i \lambda_i w_i^T \quad \Leftarrow \text{ expanded sum form}$$

$$= \begin{bmatrix} v_n & \cdots & v_1 \end{bmatrix} \begin{bmatrix} \lambda_n & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix} \begin{bmatrix} - & w_n^T & - \\ & \vdots & \\ - & w_1^T & - \end{bmatrix}$$

Coding: eig(A) => evals, evecs

Complex Eigenvalues & Eigenvectors

complex # : related to rotations

what is an eigenvector of a rotation matrix?

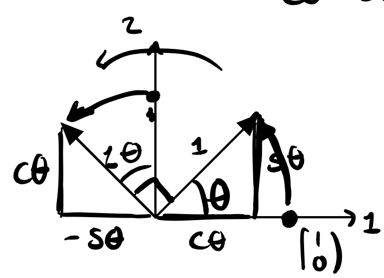
Ex. 2x2 rotation $R \in \mathbb{R}^{2\times 2}$

$c\theta^2 + s\theta^2 = 1$

$$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c\theta \\ s\theta \end{bmatrix}$$

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s\theta \\ c\theta \end{bmatrix}$$



$$\begin{bmatrix} c\theta & s\theta \end{bmatrix}\begin{bmatrix} -s\theta \\ c\theta \end{bmatrix} = 0$$

no **real** eigenvectors...

does have 2 complex eigenvectors

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} c\theta - s\theta i \\ s\theta + c\theta i \end{bmatrix} = \begin{bmatrix} c\theta - s\theta i \\ (s\theta + c\theta i)(-i)i \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} c\theta - s\theta i \\ (c\theta - s\theta i)i \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}(c\theta - s\theta i)$$

$\begin{bmatrix} 1 \\ i \end{bmatrix}$: eigenvector

$c\theta - s\theta i$: eigenvalue

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} c\theta + s\theta i \\ s\theta - c\theta i \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}(c\theta + s\theta i)$$

$\begin{bmatrix} 1 \\ -i \end{bmatrix}$: eigenvector

$c\theta + s\theta i$: eigenvalue

Diagonalize...

diagonalization of a rotation

$$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\begin{bmatrix} c\theta - s\theta i & 0 \\ 0 & c\theta + s\theta i \end{bmatrix}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \mathcal{U} \quad D \quad \mathcal{U}^{-1}$$

$$\mathcal{U}^{-1} = \mathcal{U}^*$$

$A \in \mathbb{C}^{n\times n}$ is unitary if

$A^{-1} = A^* \Leftrightarrow A^*A = I$

analogous to saying

$A^{-1} = A^T$ for $A \in \mathbb{R}^{n\times n}$

Claim: $\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ is unitary

Proof:

$$\mathcal{U}^*\mathcal{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} 1-i^2 & 1+i^2 \\ 1+i^2 & 1-i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{2} = I$$

($\mathcal{U}^*$: almost $\mathcal{U}^{-1}$)

A^*: conjugate transpose of A.

- take the transpose
- conjugate all complex #'s (negate i)

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c\theta - s\theta i & 0 \\ 0 & c\theta + s\theta i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

right evecs ✓ are conjugates of ea. other

eigen values come in conjugate pairs

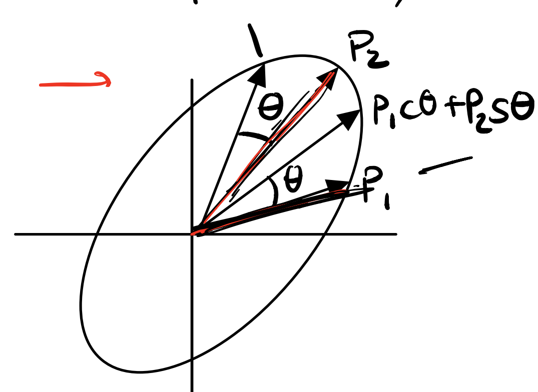
left evecs are conjugate pairs

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}}$$

Noncircular Rotations

$$\begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

$$\begin{bmatrix} P_1 c\theta + P_2 s\theta & -P_1 s\theta + P_2 c\theta \end{bmatrix}$$



$$[A]x = \left[\begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix}^{-1} \right] x$$

Convert back to the original coords

apply rotation in the P_1, P_2 coords

x written w.r.t. basis $P_1 P_2$

Better: $x = [p_1\, p_2]\, x'$ $y = [p_1\, p_2]\, y'$

A is a rotation in the x' coords ...

$$y' = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} x'$$

$$[p_1\, p_2]^{-1} y = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} [p_1\, p_2]^{-1} x$$

$$y = \underbrace{[p_1\, p_2] \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} [p_1\, p_2]^{-1}}_{A} x$$

if $A = [p_1\, p_2] \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} [p_1\, p_2]^{-1}$

and we diagonalize A ...

$$\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} [p_1\, p_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = [p_1\, p_2] \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c\theta - s\theta i & 0 \\ 0 & c\theta + s\theta i \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \frac{1}{\sqrt{2}} [p_1\, p_2]^{-1}$$

eigen values of A

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} p_1 + p_2 i & p_1 - p_2 i \end{bmatrix} \begin{bmatrix} c\theta - s\theta i & 0 \\ 0 & c\theta + s\theta i \end{bmatrix} \begin{bmatrix} q_1^T - i q_2^T \\ q_1^T + i q_2^T \end{bmatrix} \frac{1}{\sqrt{2}}$$

(with $[p_1\, p_2]^{-1} = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}$)

right evecs conjugate pairs

left evecs conjugate pairs.

$$v_1 = (p_1 + p_2 i)\frac{1}{\sqrt{2}} \qquad w_1^T = (q_1^T - i q_2^T)\frac{1}{\sqrt{2}}$$

$$v_2 = (p_1 - p_2 i)\frac{1}{\sqrt{2}} \qquad w_2^T = (q_1^T + i q_2^T)\frac{1}{\sqrt{2}}$$

General matrices w/ complex eigenvalues:

$A \in \mathbb{R}^{n \times n}$

- complex eigenvalues come in conjugate pairs.
- " eigenvectors " " " "

Diagonalization:

$$A = \underbrace{\begin{bmatrix} \begin{vmatrix} v_1 & v_2 \end{vmatrix} & \begin{vmatrix} v_3 & v_4 \end{vmatrix} & \cdots \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \begin{bmatrix} a_1 - b_1 i & 0 \\ 0 & a_1 + b_1 i \end{bmatrix} & & 0 & \cdots \\ & 0 & \begin{bmatrix} a_2 - b_2 i & 0 \\ 0 & a_2 + b_2 i \end{bmatrix} & \\ & \vdots & & \ddots \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \begin{vmatrix} v_1 & v_2 \end{vmatrix} & \begin{vmatrix} v_3 & v_4 \end{vmatrix} & \cdots \end{bmatrix}^{-1}}_{P^{-1}}$$

(v_1, v_2: conjugate pairs; v_3, v_4: conj. pairs)

$$= \begin{bmatrix} \begin{vmatrix} v_1 & v_2 \end{vmatrix} & \begin{vmatrix} v_3 & v_4 \end{vmatrix} & \cdots \end{bmatrix} \begin{bmatrix} r_1 \begin{bmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{i\theta_1} \end{bmatrix} & & 0 & \cdots \\ & 0 & r_2 \begin{bmatrix} e^{-i\theta_2} & 0 \\ 0 & e^{-i\theta_2} \end{bmatrix} & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \begin{vmatrix} v_1 & v_2 \end{vmatrix} & \begin{vmatrix} v_3 & v_4 \end{vmatrix} & \cdots \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \begin{vmatrix} p_1 & p_2 \end{vmatrix} & \begin{vmatrix} p_3 & p_4 \end{vmatrix} & \cdots \end{bmatrix} \begin{bmatrix} r_1 \begin{bmatrix} c\theta_1 & -s\theta_1 \\ s\theta_1 & c\theta_1 \end{bmatrix} & & 0 & \cdots \\ & 0 & r_2 \begin{bmatrix} c\theta_2 & -s\theta_2 \\ s\theta & c\theta_2 \end{bmatrix} & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \begin{vmatrix} p_1 & p_2 \end{vmatrix} & \begin{vmatrix} p_3 & p_4 \end{vmatrix} & \cdots \end{bmatrix}^{-1}$$

real vectors:

$p_1 = \frac{1}{\sqrt{2}}(v_1 + v_2)$ ← real part of v_1, v_2

$p_2 = \frac{-i}{\sqrt{2}}(v_1 - v_2)$ ← imag part of v_1, v_2

} → defines plane of rotation

$p_3 = \frac{1}{\sqrt{2}}(v_3 + v_4)$

$p_4 = \frac{1}{\sqrt{2}}(v_3 + v_4)$

} → defines plane of rotation

$$= \begin{bmatrix} \begin{vmatrix} p_1 & p_2 \end{vmatrix} & \begin{vmatrix} p_3 & p_4 \end{vmatrix} & \cdots \end{bmatrix} \begin{bmatrix} \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} & & 0 & \cdots \\ & 0 & \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} & \\ & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \begin{vmatrix} p_1 & p_2 \end{vmatrix} & \begin{vmatrix} p_3 & p_4 \end{vmatrix} & \cdots \end{bmatrix}^{-1}$$

Types of Matrices

Diagonal matrices : eigenvalues are the elements on the diagonal

evecs $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots$

Upper/Lower triangular matrix → evals are diagonal elements

$$\begin{bmatrix}\lambda_1 & * & \cdots & * \\ & \ddots & & \\ 0 & & & \lambda_n^*\end{bmatrix}$$ evecs: $\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$ $\begin{bmatrix}*\\1\\0\\\vdots\\0\end{bmatrix}$ $\begin{bmatrix}*\\*\\1\\0\\\vdots\\0\end{bmatrix}$

Symmetric Matrix —

$S \in \mathbb{R}^{n \times n}$ $\quad S = S^T$ $\quad$ evals: all real

evecs: all orthogonal

$\Rightarrow \exists\, R$: orthonormal $\quad R^T R = I$

s.t. $\quad S = R D R^T \Leftarrow$

Skew Symmetric Matrix

$K \in \mathbb{R}^{n \times n}$ $\quad K = -K^T$ $\quad$ evals: purely imaginary

evecs: orthogonal to ea. other.

Note: sym matrices $\Longleftrightarrow$ real numbers

skew sym matrices $\Longleftrightarrow$ imag numbers

General matrices $\Longleftrightarrow$ complex numbers

Every matrix $A \in \mathbb{R}^{n \times n}$

$$A = \underset{\text{sym}}{S} + \underset{\text{skew sym}}{K} \qquad z = a + bi$$

$$= \underbrace{\frac{1}{2}(A + A^T)}_{\text{sym}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew sym}}$$

Repeated eigenvalues can be weird...

Nilpotent Matrix $N^k = 0 \quad N^{k-1} \neq 0$ ✓

Ex. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(QNQ^{-1})^k = QNQ^{-1}QNQ^{-1}\cdots QNQ^{-1}$

$= QN^kQ^{-1} = Q0Q^{-1} = 0$

$Nv = \lambda v \Rightarrow N^k v = \lambda^k v$

$0v = \lambda^k v \quad \lambda = 0$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ only has a 1D nullspace

$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

for diagonalization... $A \in \mathbb{R}^{n \times n}$ we need a basis of eigenvectors

$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^{-1}$

basis for $\mathbb{R}^n$

if A doesn't have a basis of eigenvectors
A is not diagonalizable

Ex. $A = \lambda I + N$ ✓

eigenvalue is λ

$(\lambda I - A)v = 0 \Rightarrow -Nv = 0$

JORDAN FORM OF A MATRIX. (NOT ON AN EXAM)

any $A \in \mathbb{C}^{n\times n}$ or $A \in \mathbb{R}^{n\times n}$

$$A = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}$$

Jordan blocks; $v_1 \cdots v_n$: generalized right eigen vectors; rows are generalized left eigen vectors

$$J_i = \lambda_i I + N_i \qquad N_i = \begin{bmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}$$

Ex.

$$A = \begin{bmatrix} v_1 & v_1' & v_1'' \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} v_1 & v_1' & v_1'' \end{bmatrix}^{-1}$$

$$A\begin{bmatrix} v_1 & v_1' & v_1'' \end{bmatrix} = \begin{bmatrix} v_1 & v_1' & v_1'' \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

$Av_1 = \lambda_1 v_1$ → eigen vector

$Av_1' = v_1 + \lambda_1 v_1'$ → gen. eigen vector of order 1

$Av_1'' = v_1' + \lambda_1 v_1''$ → gen eigen vector of order 2

---

to sum up...
even if you can't find a basis of regular eigenvectors
you can find a basis of generalized eigenvectors
and the matrix can be put in Jordan form

$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n \Longrightarrow$ diagonalizable

Differential Eqns / Dynamics

"how does a system change w/ time"

autonomous: $\dot{x} = \frac{dx}{dt} = f(x)$

$x \in \mathbb{R}^n$: state

$f: \mathbb{R}^n \to \mathbb{R}^n$: dynamics

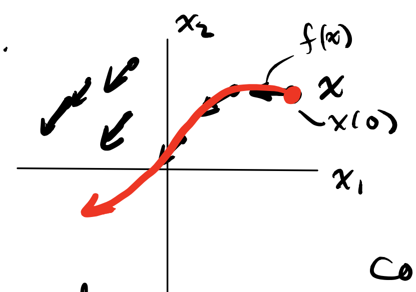
general: $\dot{x} = f(x, u)$

$u(t) \in \mathbb{R}^m$: control input

"vector field"

how to pick $u(t)$ is control theory

Ex.



Solve dynamics → computing a trajectory
"integrating a trajectory"

$x(0)$: initial condition

$x(t)$: state at time t ⟸ want to find

Linear Dynamical Systems

LINEAR TIME VARYING (LTV)

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$A(t) \in \mathbb{R}^{n \times n}$ $B(t) \in \mathbb{R}^{n \times m}$

LINEAR TIME INVARIANT (LTI)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

A: state matrix

B: input matrix

C: output matrix

Linear: linear in x & u

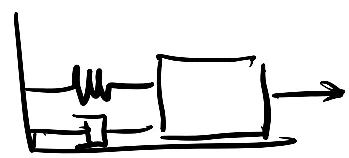
Single input system: $B \in \mathbb{R}^{n \times 1}$ $u(t) \in \mathbb{R}$
Single output system: $C \in \mathbb{R}^{1 \times n}$ $y(t) \in \mathbb{R}$

Output: $\dot{\underline{x}} = f(x, u)$

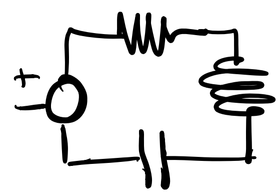
$y = \underline{h}(x, u)$ output equation

measurement (usually not the whole state) — y
state — x

Ex: spring-mass-damper



resistor-inductor-capacitor circuit



$$\boxed{\dot{x} = Ax} \Leftarrow$$

Ex. scalar $x \in \mathbb{R}$

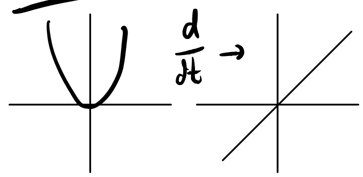
$\rightarrow \underline{\dot{x} = \lambda x} \Rightarrow$ solution

$x(t) = e^{\lambda t} x(0)$ ✓ $x(0) = x^0$

Check: $\dot{x} = \frac{d}{dt}x = \frac{d}{dt} e^{\lambda t} x(0) = \lambda e^{\lambda t} x(0) = \lambda x(t)$

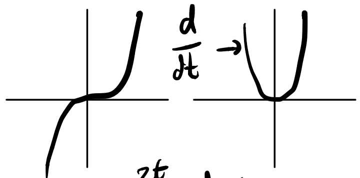
$$\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t} \Leftarrow \boxed{\frac{d}{dt} e^{t} = e^{t}}$$

Aside



$$\frac{d}{dt} t^k = k t^{k-1}$$

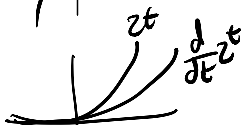
grows slower w/ t than t^k



$\frac{d}{dt} 2^t$ → grows slower than 2^t

↑

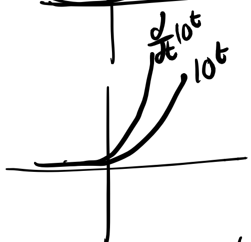
$\frac{d}{dt} e^t$ ↔ grows at the same rate as e^t

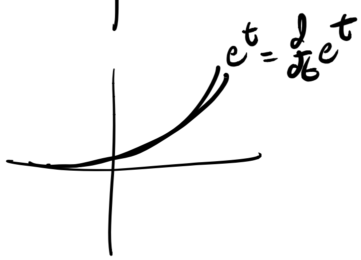


$\frac{d}{dt} 10^t$

↓

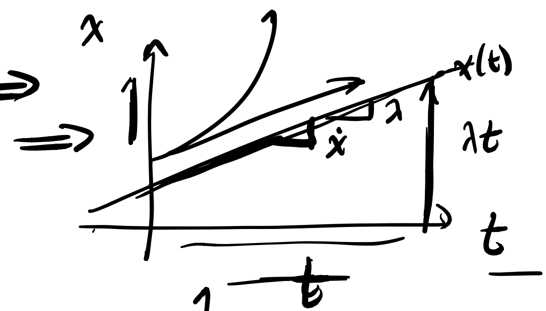
grows faster w/ t than 10^t





$$\dot{x} = \lambda x \implies x(t) = \lambda t \, x(0)$$

$$\dot{x} = \lambda \implies$$



$$\dot{x} = \lambda x \quad x(0) = x^0 \implies x(t) = e^{\lambda t} x^0$$

Vector version

$$\dot{x} = Ax \quad x(0) = x^0 \implies \boxed{x(t) = e^{At} x(0)}$$

Matrix Exponential:

$$e^{At} := I + At + \frac{A^2 t^2}{2} + \frac{1}{3!} A^3 t^3 + \cdots$$

$$= \sum_{k=0}^{\infty} (At)^k \frac{1}{k!}$$

$$\frac{d}{dt} e^{At} = A e^{At}$$

$$= \frac{d}{dt}\left(\cancelto{0}{I} + At + \frac{A^2t^2}{2} + \frac{1}{3!}A^3t^3\right)$$

$$A + A(At) + \frac{1}{2}A(At)^2 + \cdots$$

$$A\left(I + At + \frac{1}{2}(At)^2 + \cdots\right)$$

$$A = PDP^{-1}$$

$$\Rightarrow e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$