Stability:

LTI: $\dot{x} = Ax + Bu$
(state, control input)

$\dot{x} = \frac{dx}{dt}$

LTI: Autonomous $(u=0)$

$\dot{x}(t) = Ax(t) \qquad x(0) = x^0$

stability a property of this matrix

does $x(t)$ blow up or decrease

Preview: feedback control

$\dot{x} = Ax + Bu$ set $u = Kx$

gain matrix

state

control input function of state

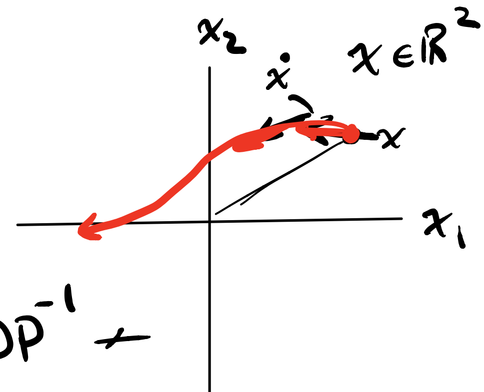
$= Ax + BKx$

$= (A + BK)x$

get to pick

Vector Case:

$A \in \mathbb{R}^{n \times n} \quad x \in \mathbb{R}^n$



→ $\dot{x} = Ax$

diagonalize $A = PDP^{-1}$

cols are right evecs

$x = Pz$

coords w.r.t. eigenvectors

$\dot{x} = P\dot{z}$

$$\dot{x} = PDP^{-1}Pz$$

$$P^{-1}\dot{x} = Dz$$

$$\dot{z} = Dz$$

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \dot{z}_n \end{bmatrix}$$

$$\left.\begin{matrix} \left[\, \dot{z}_1 = \lambda_1 z_1 \right. \\ \vdots \\ \left[\, \dot{z}_n = \lambda_n z_n \right. \end{matrix}\right\} \rightarrow$$ coords w.r.t. eigen vectors evolve separately from each other

by analyzing behavior of system in ea. eigen subspace (based on the eigenvalue) we can understand overall behavior.

Scalar case:

$$\dot{z}_k = \lambda_k z_k \quad z_k(0) = z_k^0 \implies z_k(t) = e^{\lambda_k t} z_k(0)$$

$$\frac{d}{dt} z_k(t) = \frac{d}{dt} e^{\lambda_k t} z_k(0) = \lambda_k \underbrace{e^{\lambda_k t} z_k(0)}_{z_k(t)}$$

$$\dot{z}_i = \lambda_i z_i$$

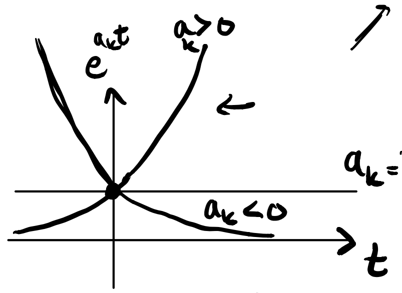
$$\lambda_k = a_k + b_k i = (a_k + b_k j)$$

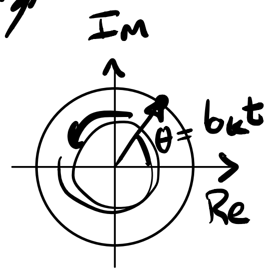
Python

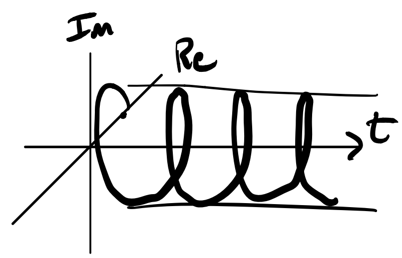
$1+1j$

$$e^{\lambda_k t} = e^{(a_k + b_k i)t} = e^{a_k t}\left(e^{b_k i t}\right)$$

$$\lambda = re^{i\theta}$$







$Re(\lambda_k)$: explodes or decays

Stability (continuous Time) $\dot{x} = Ax$

- **Stable**: $\mathrm{Re}(\lambda_k) < 0$ $k = 1, \dots, n$]
- **Marginally stable**: $\mathrm{Re}(\lambda_k) = 0$ for some k
  (others $\mathrm{Re}(\cdot)$
- **Unstable** $\mathrm{Re}(\lambda_k) > 0$ for any k]

Complex Plane:

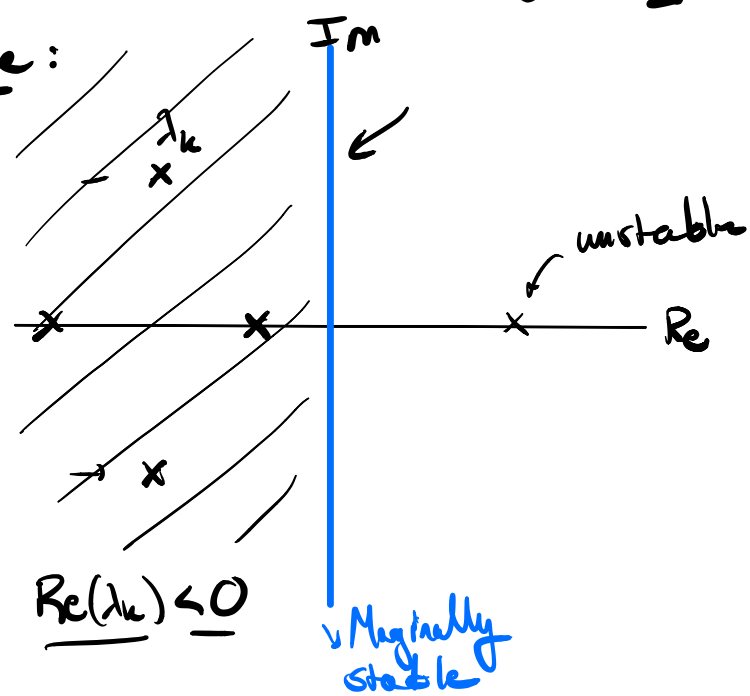
$A \in \mathbb{R}^{n \times n}$

A is stable

iff $\rho(A) \subset \mathrm{OLHP}$

(spectrum of A)

subset



Solution to

$$\left[\ \dot{x} = Ax \quad x(0) = x^0 \right. \Rightarrow \boxed{x(t) = e^{At} x(0)}$$

Matrix Exponential:

$$e^{At} := I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} (At)^k \frac{1}{k!}$$

$$\frac{d}{dt} e^{At} = A e^{At}$$ taylor expansion of $e^{(\cdot)}$

$$\frac{d}{dt}\left(I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!}\right) = A + A^2 \frac{2t}{2} + A^3 \frac{3t^2}{3!}$$

$\frac{d}{dt} t^k \to k t^{k-1}$

$$= A\left(I + At + A^2 \frac{t^2}{2!} + \cdots\right.$$

$$e^{At} = P e^{Dt} P^{-1} = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$x(t) = e^{At} x(0) = P \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \underbrace{P^{-1} x(0)}_{z(0)}$$

$z(t) = P^{-1} x(t) \ldots$

$\boxed{e^{At} \in \mathbb{R}^{n \times n} : \text{ state transition matrix (for LTI dynamics)}}$

$$x(t) = \left[e^{At} \right] x(0)$$

Note:

$\longrightarrow$ $e^{At} = I + At + \frac{A^2 t^2}{2} + \frac{1}{3!} A^3 t^3 + \cdots$ $\longrightarrow$

Cayley Hamilton: $A \in \mathbb{R}^{n \times n}$

$$\Rightarrow A^P = \beta_{n-1} A^{n-1} + \beta_{n-2} A^{n-2} + \cdots + \beta_1 A^1 + \beta_0 I$$

$$e^{At} = \beta_{n-1}(t)A^{n-1} + \beta_{n-2}(t)A^{n-2} + \cdots + \beta_1(t)A + \beta_0(t)I$$

for some $\beta_k(t)$...

Homework 3: prob 6 (reference)

Discrete Time Dynamics

(before) $\underline{\dot{x}} = A\underline{x} + Bu$ (continuous time)

differential eqn...

(discrete time)

t : was a continuous variable

define a time step Δt ...

" " " index t' ...

$\underline{t = t'\Delta t}$ t' : # of time steps

update equation

$$\Rightarrow \underbrace{x(t'+1)}_{\text{state at the next time step}} = A'\underbrace{x(t')}_{\text{state}} + B'u(t') \Leftarrow \text{good for computation}$$

usually just write

$$\left[\, x(\underline{t+1}) = A\,x(\underline{t}) + Bu(t) \,\right] \leftarrow$$

Given $\underline{\dot{x}} = \underline{Ax + Bu} \Leftarrow$ cont. time

how do we compute $x(t'+1) = \underline{A'}x(t') + B'u(t')$?

assume $u(t') = 0$: will relax later...

given $\dot{x}(t) = Ax(t)$

find A' such that $x(t'+1) = A'x(t')$

$$t = t'\Delta t$$

$$x(t) = e^{At}x(0)$$

$$x((t'+1)\Delta t) = e^{A((t'+1)\Delta t)}x(0)$$

$$= e^{A\Delta t}\underbrace{e^{A(t'\Delta t)}x(0)}_{e^{At}x(0)}$$

$$x(t'+1) = \underbrace{e^{A\Delta t}}_{A'}x(t')$$

$$x(t'+1) = A'x(t')$$

$$\rightarrow \boxed{A' = \underbrace{e^{A\Delta t}}}$$

how x evolves over the time interval Δt

Stability in Discrete Time:

$$\dot{x} = Ax \qquad x^+ = A'x \qquad A' = e^{A\Delta t}$$

stability $\Rightarrow$ if $\lambda \in spec(A)$

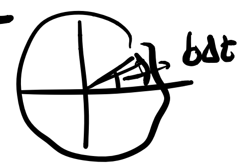
$\lambda \in spec(A)$ $Re(\lambda) < 0$ $\quad$ $\mu = e^{\lambda\Delta t} \in spec(A')$ $\quad$ Spectral mapping thm..

$$\text{if } Re(\lambda) < 0 \Leftrightarrow |\mu| = |e^{\lambda\Delta t}| < 1$$

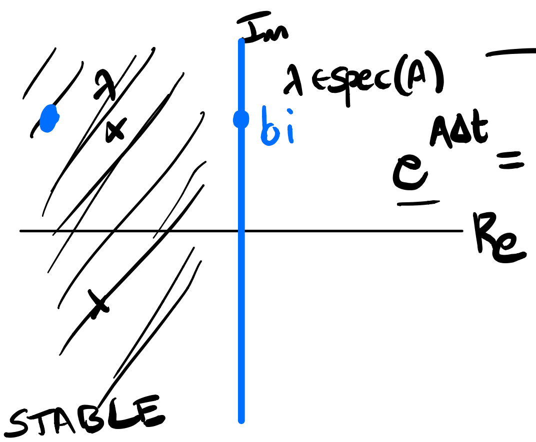
$$\lambda = a + bi \qquad |\mu| = |e^{\lambda \Delta t}| = |e^{(a+bi)\Delta t}|$$
$$= |e^{a\Delta t} e^{bi\Delta t}|$$
$$= |e^{a\Delta t}|\underbrace{|e^{bi\Delta t}|}_{1}$$
$$= |e^{a\Delta t}|$$

if $a < 0$

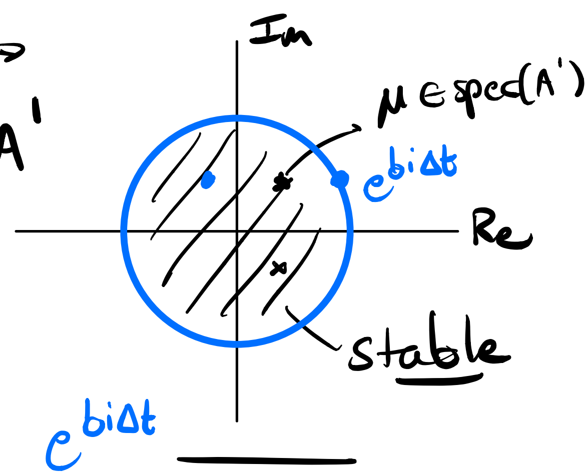
$$|e^{a\Delta t}| < 1$$



Continuous Time: **Discrete Time**



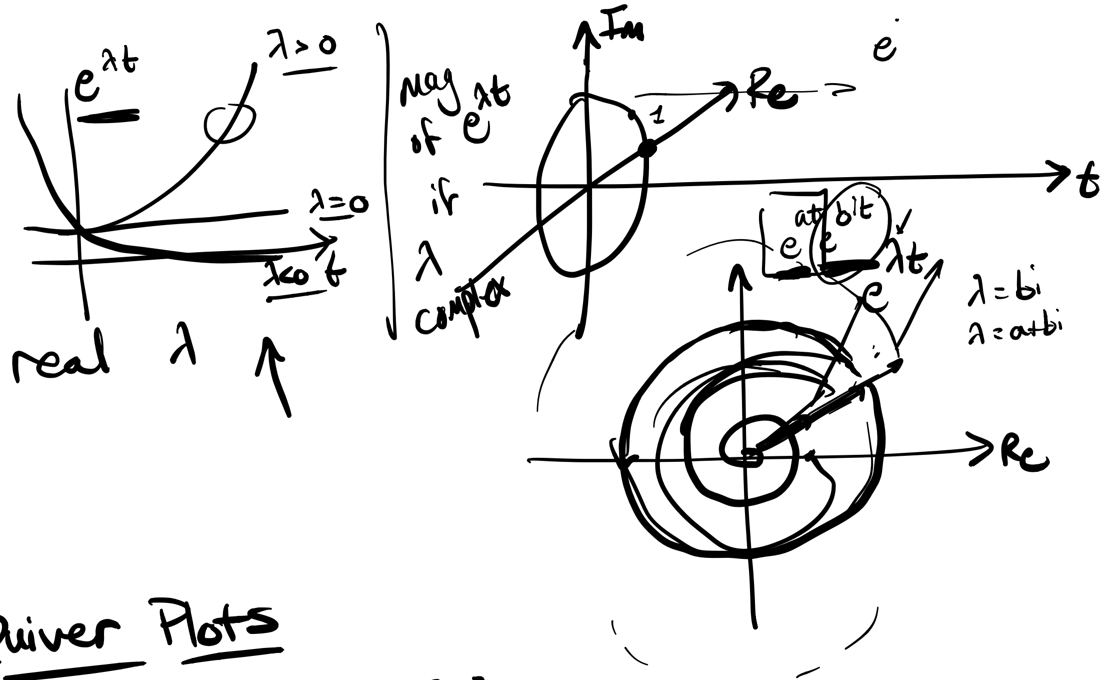
$$e^{A\Delta t} = A'$$



Stability (Discrete time)

$$x^+ = A'x \qquad \mu \in \mathrm{spec}(A')$$

- stable: $|\mu| < 1$ for all $\mu \in \mathrm{spec}(A')$
- marg. stable: $|\mu| = 1$ for some μ
- unstable: $|\mu| > 1$ for some μ



Quiver Plots

$\dot{\underline{x}} = A x \quad A \in \mathbb{R}^{2\times 2}$

eigenvalues of 2x2 matrices ...

$\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(sI - A) = 0$

$\det \begin{vmatrix} s-a & -b \\ -c & s-d \end{vmatrix} = 0$

$(s-a)(s-d) - bc = 0$

$s^2 - (a+d)s + ad - bc = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$s^2 - \mathrm{Tr}(A)s + \det(A) = 0$

quadratic eqn

$$\boxed{\lambda_{1,2} = \frac{Tr(A)}{2} \pm \sqrt{\underbrace{\left(\frac{Tr(A)}{2}\right)^2 - det(A)}}}$$

average diagonal element

sloppy

if $det(A) = 0$ $\quad \frac{Tr(A)}{2} \pm \sqrt{\left(\frac{Tr(A)}{2}\right)^2}$

$$\lambda_{1,2} = Tr(A),\ 0$$

if $0 < det(A) < \left(\frac{Tr(A)}{2}\right)^2$ $\quad \sqrt{\left(\frac{Tr(A)}{2}\right)^2 - det(A)}$

$$\lambda_{1,2} = \frac{Tr(A)}{2} \pm \left(\sqrt{\underbrace{\qquad\qquad}}\right)$$

smaller in mag than $\frac{Tr(A)}{2}$

$\lambda_{1,2} > 0$
real

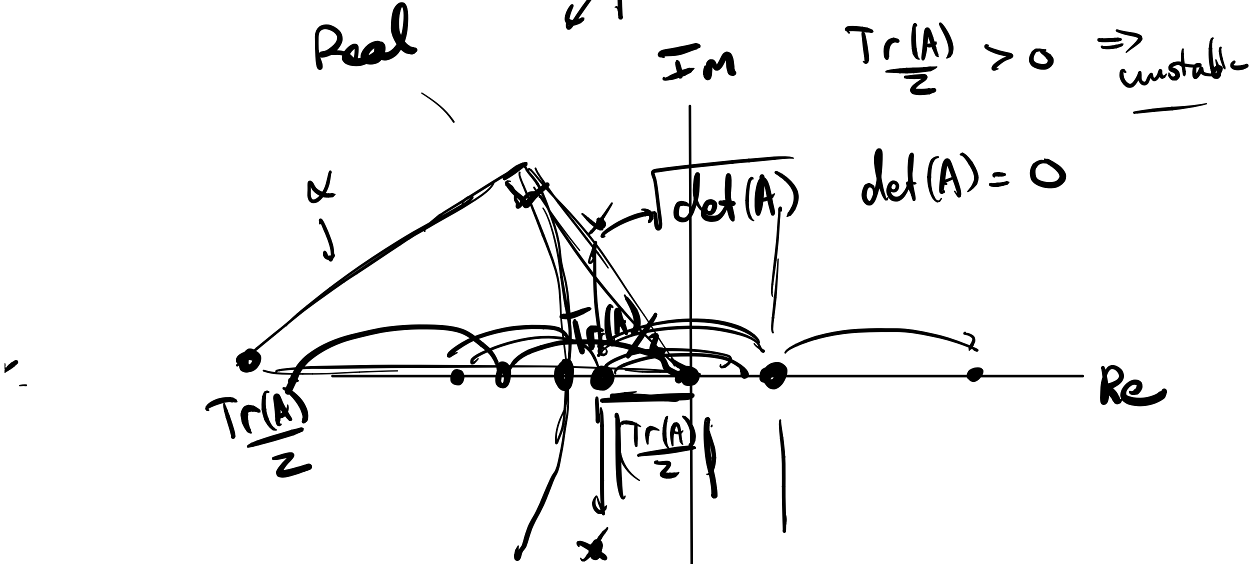
$$det(A) > \left(\frac{Tr(A)}{2}\right)^2 \Rightarrow \sqrt{\underbrace{\left(\frac{Tr(A)}{2}\right)^2 - det(A)}}$$

$$\Rightarrow \lambda_{1,2} = \frac{Tr(A)}{2} \pm (\text{imaginary})^{<0}$$

$$QAQ^{-1} \sim \begin{vmatrix} \boxtimes & \\ & \boxtimes \end{vmatrix}$$

$A \in \mathbb{R}^{2\times 2}$

$$\boxed{\lambda_{1,2} = \frac{Tr(A)}{2} \pm \sqrt{\left(\frac{Tr(A)}{2}\right)^2 - \det(A)}}$$



$$\left(\frac{Tr(A)}{2}\right)^2 - \det(A) = dsc \leftarrow \text{discriminant}$$

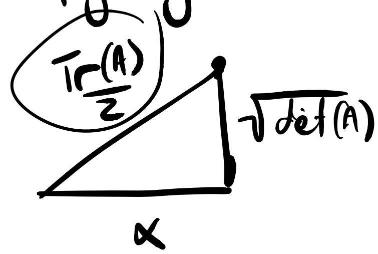
$$\lambda_{1,2} = \frac{Tr(A)}{2} \pm \alpha$$

where $\alpha^2 = \left(\frac{Tr(A)}{2}\right)^2 - \det(A)$

$$\alpha^2 + \det(A) = \left(\frac{Tr(A)}{2}\right)^2$$

$$\alpha^2 + \left(\sqrt{\det(A)}\right)^2 = \left(\frac{Tr(A)}{2}\right)^2$$

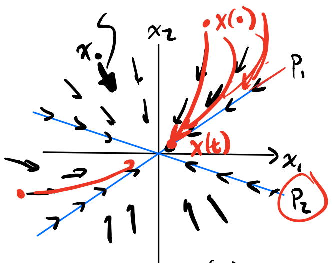
Pythagoras...



Quiver Plots $A \in \mathbb{R}^{2\times 2}$ $\dot{x} = Ax$ $x \in \mathbb{R}^2$

$\lambda_{1,2} \in \text{spec}(A)$

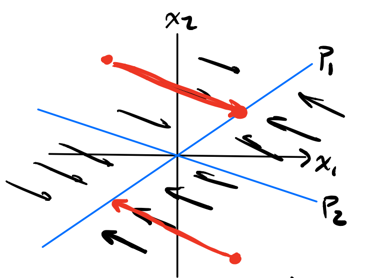
$\dot{x} = Ax$



$Re(\lambda_1) < 0$ $Re(\lambda_2) < 0$

$Im(\lambda_1) = Im(\lambda_2) = 0$

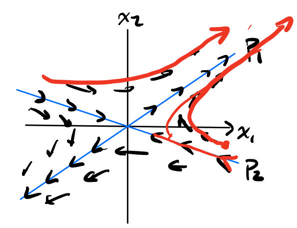
stable



$Re(\lambda_1) = 0$ $Re(\lambda_2) < 0$

$Im(\lambda_1) = Im(\lambda_2) = 0$

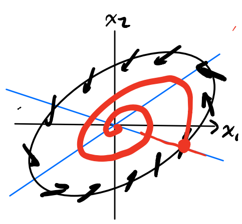
marg stable



$Re(\lambda_1) > 0$ $Re(\lambda_2) < 0$

$Im(\lambda_1) = Im(\lambda_2) = 0$

unstable

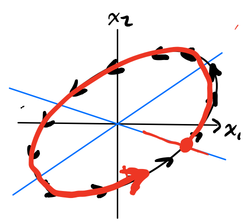


$\lambda_1 = a+bi$ $\lambda_2 = a-bi$

$b \neq 0$

$a < 0$

stable

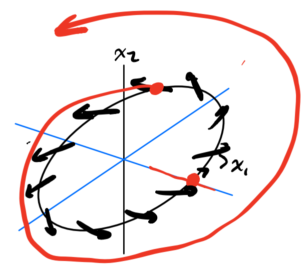


$\lambda_1 = a+bi$ $\lambda_2 = a-bi$

$b \neq 0$

$a = 0$

Marg. stable



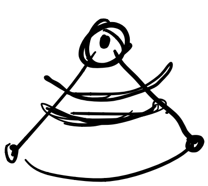
$\lambda_1 = a+bi$ $\lambda_2 = a-bi$

$b \neq 0$

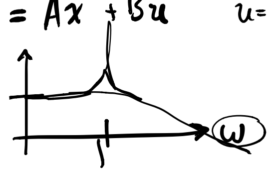
$a > 0$

unstable

No damping



$\rightarrow \dot{x} = Ax + Bu$ $u = \sin(\omega t)$

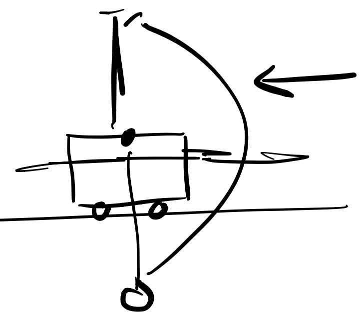


$\lambda = a+bi$

$g\cos(\theta)$

$$\dot{x} = Ax + Bu \qquad u = Kx$$

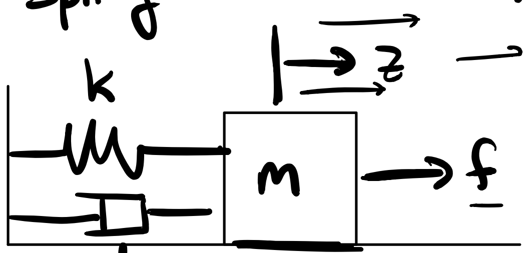
$$\dot{x} = \underbrace{(A + BK)}_{\text{state matrix } \bar{w} \text{ feedback}} x$$



Examples of Dynamics

2 exs.
- spring – mass – damper
- resistor – inductor – capacitor (RLC circuits)

1. Spring - mass - damper



k: spring constant

$f_k = kz$

d: damping coeff

$f_d = d\dot{z}$

m: mass

$\Sigma f = m\ddot{z}$

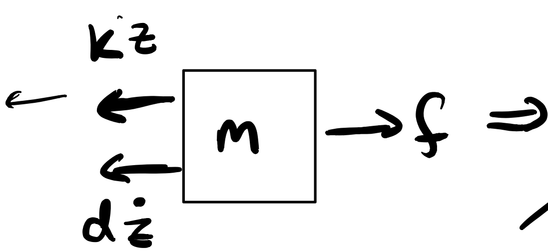
$(f = ma)$

Newton's 2nd law

f: force

u

Free Body Diagram



$\Rightarrow$

$$\begin{cases} m\ddot{z} = f - kz - d\dot{z} \\ \ddot{z} = \dfrac{f}{m} - \dfrac{k}{m}z - \dfrac{d}{m}\dot{z} \end{cases}$$

$\downarrow$

$$\dot{x} = Ax + Bu$$

2nd order system: physics

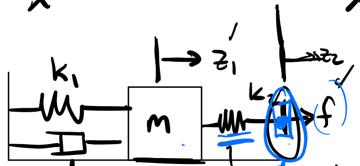
$\dot z = \cancel{\ }$ ______

$\ddot z =$ ______

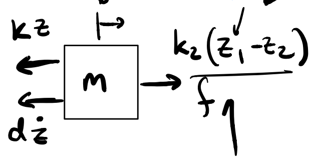
$$x = \begin{bmatrix} z \\ \dot z \end{bmatrix} \begin{array}{l} \leftarrow \text{position} \\ \leftarrow \text{velocity} \end{array}$$

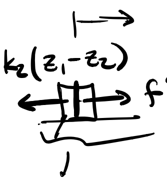
$$\dot x = \begin{bmatrix} \dot z \\ \ddot z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} z \\ \dot z \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \quad (u = f)$$

$$\dot x = A x + B u$$



1 block/mass
↘ position
velocity



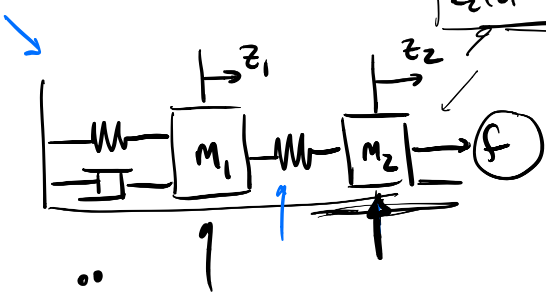


$m\ddot z_1 = \Sigma f$

$0\ddot z_2 = \Sigma f$

$0 = \Sigma f$

$$\boxed{k_2(z_1-z_2) = f}$$

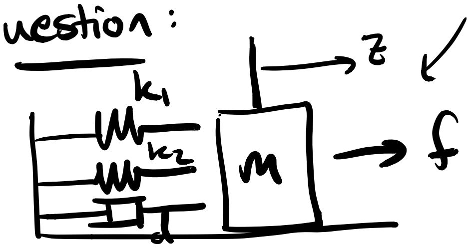


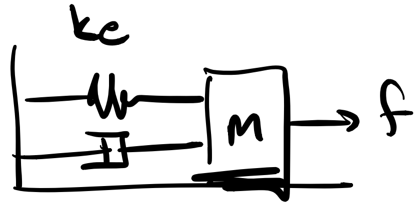
$m_1\ddot z_1 =$ ______ $\qquad m_2\ddot z_2 =$ ______

$$x = \begin{bmatrix} z_1 \\ \dot z_1 \\ z_2 \\ \dot z_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot z_1 \\ \ddot z_1 \\ \dot z_2 \\ \ddot z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \multicolumn{4}{c}{\text{first block}} \\ 0 & 0 & 0 & 1 \\ \multicolumn{4}{c}{\text{2nd block}} \end{bmatrix} \begin{array}{c} z_1 \\ \dot z_1 \\ z_2 \\ \dot z_2 \end{array} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \end{bmatrix} u$$

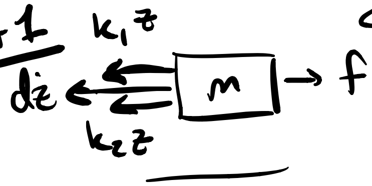
2 blocks/masses → 2 positions
2 velocities

Question:





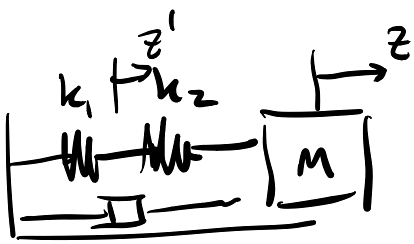
opt 1

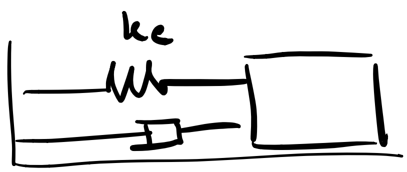


opt 2

springs in parallel

$\to k_e = k_1 + k_2$

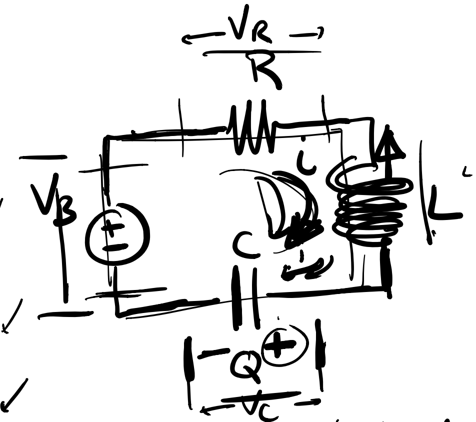




springs in series

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

RLC:



Voltage rules

$$V_R = Ri = R\dot{Q}$$
$$V_L = L\frac{di}{dt} = L\ddot{Q}$$
$$V_C = QC = CQ$$

R : resistance (like damper)

L : inductance (like mass)

C : capacitance (like spring stores energy)

Q : charge on capacitor (like position)

i : current (velocity)

$$i = \frac{dQ}{dt} = \dot{Q} \qquad \dot{i} = \ddot{Q}$$

Kirchoff's Laws

Current law (KCL)

$$\boxed{\text{sum of currents in/out of junction} = 0}$$

$$i = i_1 + i_2$$

Voltage law (KVL)

$$\boxed{\text{sum of voltages drops around loops} = 0} \Rightarrow V_B = V_R + V_L + V_C$$

$$V_B = R\dot{Q} + L\ddot{Q} + CQ$$

$$x = \begin{bmatrix} Q \\ \dot{Q} \end{bmatrix} \begin{matrix} \leftarrow \text{charge} \\ \leftarrow \text{current} \end{matrix}$$

$$L\ddot{Q} = -R\dot{Q} - CQ + V_B$$

$$\ddot{Q} = -\frac{C}{L}Q - \frac{R}{L}\dot{Q} + \frac{1}{L}V_B$$

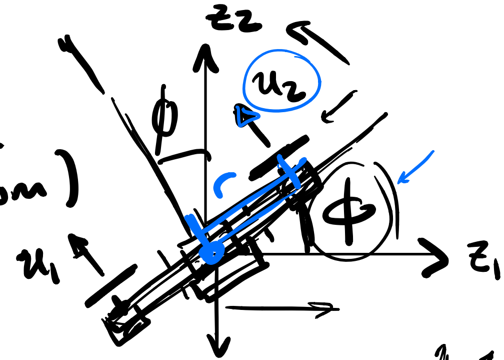
$$\underbrace{\dot{x} = \begin{bmatrix}\dot{Q}\\ \ddot{Q}\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & 1\\ -\frac{C}{L} & -\frac{R}{L}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}Q\\ \dot{Q}\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0\\ 1/L\end{bmatrix}}_{B}\underbrace{u}_{u}$$

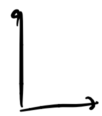
V_B

---

ea. <u>state</u> related to something w/
<u>momentum</u> / <u>inertia</u> / "<u>memory</u>"

Simple Nonlinear Quadrotor (Drone)





2D

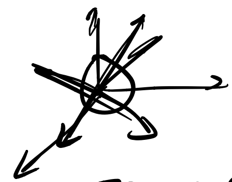
3 DoF (degrees of freedom)

↓

$x \in \mathbb{R}^6$

$$\boxed{\text{“}\Sigma f = ma\text{”}}$$

2D. (3D)



$$\begin{bmatrix} m\ddot{z}_1 = -\overbrace{(u_1+u_2)}^{w_1}\sin\phi \\ m\ddot{z}_2 = \overbrace{(u_1+u_2)}^{w_1}\cos\phi - m\underbrace{g}_{\text{gravity}} \end{bmatrix}$$

linear acceleration

→ 3D position
→ 3D orientation
→ 6 DoF degrees of freedom

$x \in \mathbb{R}^{12}$

$$\boxed{\text{“}\Sigma \tau = I\ddot{\phi}\text{”}}$$

→ 1D angular acceleration (3D)

sum of torques

moment of inertia

$$I\ddot{\phi} = ru_2 - ru_1 = r\overbrace{(u_2-u_1)}^{w_2}$$

Dynamics

$$x = \begin{bmatrix} z_1 \\ z_2 \\ \phi \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{\phi} \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1+u_2 \\ u_2-u_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\phi} \\ \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\phi} \\ -\frac{1}{m}\sin\phi\,(u_1+u_2) \\ \frac{1}{m}\cos\phi\,(u_1+u_2) - g \\ \frac{r}{I}(u_2-u_1) \end{bmatrix}$$

modeling methods

- Newtonian eqns.
- Lagrangian → energy take derivatives