Complex Evecs:

$A \in \mathbb{R}^{n\times n}$ $\qquad$ $a, b \in \mathbb{R}$

compute evals $\lambda, \bar{\lambda} = a \pm bi \qquad b \neq 0$

compute evecs $p, \bar{p} = \underline{u} \pm \underline{v}i \qquad p = u+vi$

$p, \bar{p} \in \mathbb{C}^n \qquad \underline{u}, \underline{v} \in \mathbb{R}^n$

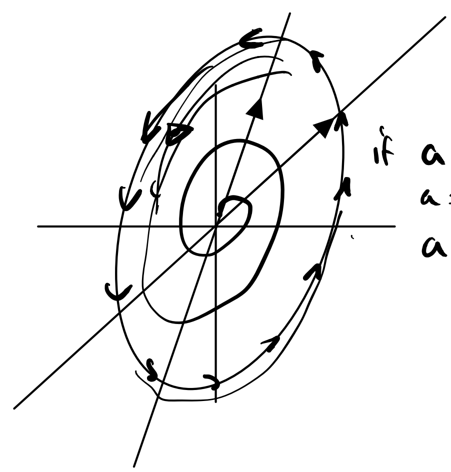
$\underline{\dot{x} = Ax}$

Ex. $p = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \quad \bar{p} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$p = u + vi$

$\bar{p} = u - vi$



if $a < 0$

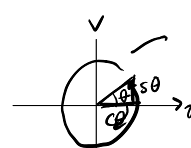
$a = 0$

$a > 0$

$$\begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \longrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} u-vi & u+vi \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} u-vi & u+vi \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}^{-1} =$$



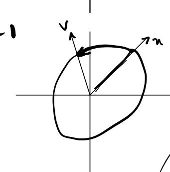
$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$ CCW

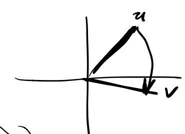
$\begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$ CW

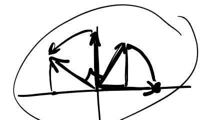
$\theta \Rightarrow -\theta$

$c(-\theta) = c\theta$

$s(-\theta) = -s\theta$







$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} u & v \end{bmatrix} r \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} u & v \end{bmatrix}^{-1}$

Coordinates Transforms on dynamics:

CT

$$\dot{x} = Ax + Bu \qquad x(0) = x^0 \qquad x \in \mathbb{R}^n$$

$$y = Cx + Du$$

Question: $x = Pz \qquad z \in \mathbb{R}^n$

$$\dot{x} = P\dot{z}$$

$$\rightarrow P\dot{z} = APz + Bu$$

$$\dot{z} = \underbrace{P^{-1}AP}_{\bar{A}}z + \underbrace{P^{-1}B}_{\bar{B}}u \rightarrow \begin{aligned} \dot{z} &= \bar{A}z + \bar{B}u \\ y &= \bar{C}z + \bar{D}u \end{aligned}$$

$$y = \underbrace{CP}_{\bar{C}}z + \underbrace{D}_{\bar{D}}u$$

DT

$$x^+ = Ax + Bu \qquad x = Pz$$

$$y = Cx + Du \qquad x^+ = Pz^+$$

$$\Longrightarrow z^+ = P^{-1}APz + P^{-1}Bu$$

$$y = CPz + Du$$

Solving ODEs w/ control inputs.

Before CT: $\dot{x} = Ax \quad x(0) = x^0 \implies x(t) = e^{At}x(0)$

DT: $x^+ = Ax \quad x(0) = x^0 \implies x(k) = A^k x(0)$

Now: $\dot{x} = Ax + Bu \quad x(0) = x^0$

$\rightarrow x^+ = Ax + Bu$

$\rightarrow$ solve?

Discrete Time:

$\rightarrow x^+ = Ax + Bu \quad x(0) = x^0$

$u(j) \leftarrow$ control input at time j

iteratively apply update equation to get solution

$x(1) = Ax^0 + Bu(0)$ first time step

$x(2) = Ax(1) + Bu(1)$ second time step...

$= A(Ax^0 + Bu(0)) + Bu(1)$

$= A^2x^0 + ABu(0) + Bu(1)$

$x(3) = Ax(2) + Bu(2)$

$= A(A^2x^0 + ABu(0) + Bu(1)) + Bu(2)$

$= A^3x^0 + A^2Bu(0) + ABu(1) + Bu(2)$

$\downarrow \quad \vdots \quad \downarrow$

$$\boxed{x(k) = A^k x^0 + \sum_{j=0}^{k-1} A^{(k-1-j)} B u(j)}$$

x at time step k

drift term

cummulative affect of control inputs

$j = 0 \Rightarrow A^{k-1} B u(0)$

$\vdots$

$j = k-1 \Rightarrow A^0 B u(k-1)$

$= B u(k-1)$

ea. term $\quad \underbrace{A^{k-1-j}}_{\text{evolution of effect of input at time } j \text{ till time } k} \; \underbrace{B u(j)}_{\text{input at time step } j}$

Continuous Time:

$$\dot{x} = Ax + Bu, \quad x(0) = x^0$$

$\dot{x} = Ax$

$\Rightarrow x(t) = e^{At} x(0)$

integrate the differential equation...

$$\boxed{x(t) = \underbrace{e^{At} x(0)}_{\text{drift term}} + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{cummulative effects of control}}}$$

τ is j from above in cont time

$\tau = 0 \Rightarrow e^{At} B u(0)$

$\tau = t \Rightarrow e^{A(t-t)} B u(t)$

$I B u(t)$

under the integral

what is $\dot{x}(t)$ if $x(t)$ is given by *

WTS: $\frac{d}{dt}x(t) = Ax(t) + Bu(t)$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right)$$

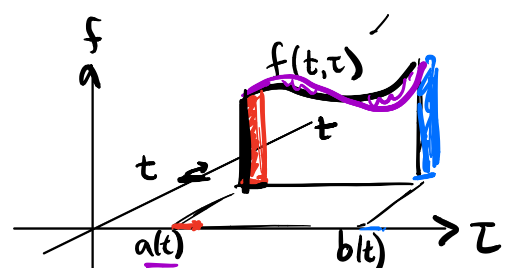
$$\rightarrow = Ae^{At}x(0) + \frac{d}{dt}\left(\int_0^t \underbrace{e^{A(t-\tau)}Bu(\tau)}_{f(t,\tau)}d\tau\right)$$

Leibnitz Integral Rule:

$\dot{x}$: Newtonian

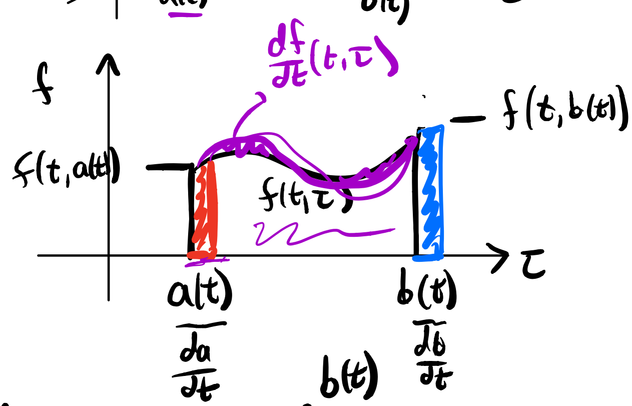
$\frac{dx}{dt}$: Leibnitz

$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(t,\tau)\,d\tau\right) =$$



$=$

$\frac{da}{dt}f(t,a(t))$



$$\rightarrow = \frac{db}{dt}f(t,b(t)) - \frac{da}{dt}f(t,a(t)) + \int_{a(t)}^{b(t)} \frac{df}{dt}(t,\tau)\,d\tau$$

$$\frac{d}{dt}x(t) = \underline{Ae^{At}x^0} + 1\,\underline{\overset{I}{\cancel{e^{A(t-t)}}}\,Bu(t)} \quad - \overset{0}{\underbrace{\qquad}}$$

$$+ \underline{\int_0^t Ae^{A(t-\tau)}Bu(\tau)\,d\tau}$$

$$= A\Big(\underbrace{e^{At}x^0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{x(t)}\Big) + Bu(t)$$

$$= Ax(t) + Bu(t)$$