Controllability

Observability

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Review of the solution to LTI systems

The continuous linear time-invariant (CLTI) system

$$\dot{x} = Ax + Bu \qquad y = Cx + Du \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

The solution to this system is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$ (A, B)

$$y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$ (A, C)

The homogeneous CLTI (H-CLTI) system

$$\dot{x} = Ax$$

The solution to this system is given by

$$x(t) = \Phi(t, t_0)x_0 = e^{A(t-t_0)}x_0$$

Review of the solution to LTI systems

The discrete linear time-invariant (DLTI) system

$$x_{k+1} = Ax_k + Bu_k \qquad y_k = Cx_k + Du_k \qquad x(t_0) = x_0 \in \mathbb{R}^n$$

The solution to this system is given by

$$x_k = A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} Bu[m]$$

$$y_k = CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-1-m} Bu[m] + Du[k]$$

The homogeneous DLTI (H-DLTI) system

$$x_{k+1} = Ax_k$$

The solution to this system is given by

$$x_{k+1} = A^k x[0]$$

Definition of controllable system

- Continuous time

$$\dot{x} = Ax + Bu$$

The state equation above or the pair (A,B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an input that transfers x_0 to x_1 in a finite time. Otherwise this state equation or (A,B) is said to be uncontrollable.

- Discrete time

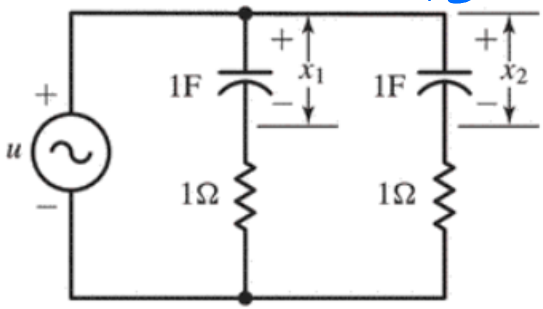
$$x_{k+1} = Ax_k + Bu_k \qquad x \in \mathbb{R}^n$$

The state equation above or the pair (A,B) is said to be controllable if for any initial state $x(0) = x_0$ and any final state x_1, there exists an *input sequence of finite length* that transfers x_0 to x_1. Otherwise this state equation or (A,B) is said to be uncontrollable.

Uncontrollable systems

Examples

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



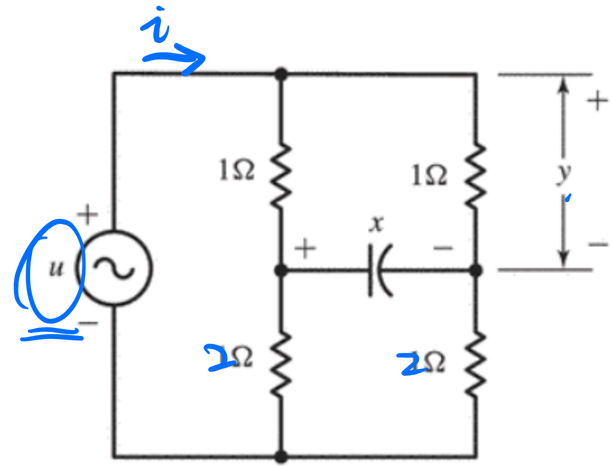
parallel RC system

if $x_1(0) = x_2(0) = 0$

then $x_1(t) = x_2(t) \quad \forall t \geq 0.$

$\underline{\underline{x_f}} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$ from $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

↑ cannot reach this state.



if $\underline{x(0) = 0}$ initial.

$\underline{\underline{x(t)}} = 0 \quad \forall t \geq 0.$

$\dot{x} = A\underline{x} + Bu.$

Test if a system is controllable

$x_{k+1} = Ax_k + Bu_k$.

Theorem (6.D1 in Chen's book)

The following statements are equivalent (iff).

1. The n-dimensional pair (A,B) is controllable.

$\Updownarrow$

2. The $n \times n$ matrix

$$W_{dc}[n-1] = \sum_{m=0}^{n-1} (A)^m BB'(A')^m \qquad \|B^T(A^T)^m\|_2^2$$

is nonsingular.

$$x_{k+1} = A^k x_0 + \sum_{m=0}^{k} A^{k-1-m} B u[m].$$

3. The $n \times np$ $\boxed{\text{controllability matrix}}$

$$C_d = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \quad n \times np \qquad B \in \mathbb{R}^{n \times p}.$$

Has rank n (full row rank).

PBH 4. The $n \times (n+p)$ matrix $[A - \lambda I \quad B]$ has full row rank at every eigenvalue, λ, of A $\forall \lambda \in \mathbb{C}$

5. If, in addition, all eigenvalues of A have magnitudes less than 1, then the unique solution of

$$W_{dc} - AW_{dc}A' = \cancel{-BB'} \quad BB' \qquad \text{Lyapunov equation.}$$

is positive definite. The solution is called the discrete Controllability Grammian and can be expressed as

$$W_{dc} = \sum_{m=0}^{\infty} A^m BB'(A)^m$$

Test if a system is controllable

Theorem (6.1 in Chen's book)

1. The n-dimensional pair (A,B) is controllable.
2. The $n \times n$ matrix

$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}d\tau$$

   is nonsingular for any $t > 0$
3. The $n \times np$ controllability matrix

$$C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

   has rank n (full row rank)
4. The $n \times (n+p)$ matrix $[A - \lambda I \quad B]$ has full row rank at every eigenvalue, λ, of A PBH
5. If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$AW_c + W_cA' = -BB'$$

   is positive definite. The solution is called the Controllability Grammian and can be expressed as

$$W_c = \int_0^\infty e^{A\tau}BB'e^{A'\tau}d\tau$$

Proof of theorem 6.1

- Controllability matrix for DLTI
- Statement 1 and 2 for CLTI
- PBH test

statement.

(A, B) is controllable $\Leftrightarrow$ rank$(C) = n$.

$C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$. $\quad A^nB \quad A^{n+1}B$. $\quad k \geq n$.

Cayley Hamilton Theorem

$$P(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 = 0.$$

$$A^n = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = 0.$$

for $k \geq n$ A^k can be expressed as linear combination of $\{A^{n-1} \; A^{n-2} \; \cdots \; A \; I\}$

$A^n B = a_{n-1} A^{n-1} B + a_{n-2} A^{n-2} B + \cdots + a_1 AB + a_0 B$

$A^k B$ can be expressed as a linear combination of $\{A^{n-1}B, \cdots AB, B\}$

for the discrete LTI $x_{k+1} = A x_k + B u_k$

$$x_{k+1} = A^{k+1} x_0 + \sum_{m=0}^{k} A^{k-1-m} B u[m]$$

at time step n.

$$x_n = A^n x_0 + \sum_{m=0}^{n-1} A^{n-1-m} B u[m]$$

x_f

$$= A^n x_0 + [A^{n-1}B \ \ A^{n-2}B \ \cdots \ AB \ \ B] \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[n-2] \\ u[n-1] \end{bmatrix}$$

$$= A^n x_0 + \underbrace{[B \ \ AB \ \cdots \ A^{n-2}B \ \ A^{n-1}B]}_{C_{n \times np}} \begin{bmatrix} u[n-1] \\ \vdots \\ u[1] \\ u[0] \end{bmatrix}$$

$$x_n - A^n x_0 = C \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}.$$

$n = \text{rank}(C)$

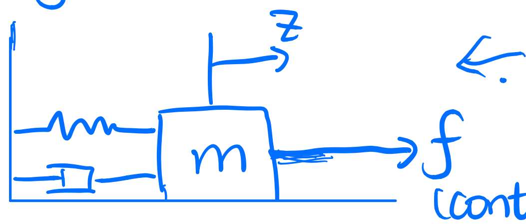
basis for $\mathbb{R}^{n \times n}$.

Application:

compute C first

then compute rank(C)

Spring - mass - damper



(control input)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \qquad n=2.$$

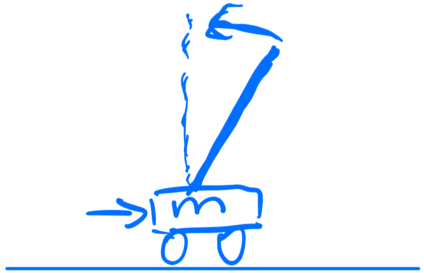
$$C = [B \ \ AB] = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{d}{m^2} \end{bmatrix} \qquad \text{rank}(C)=2.$$

this system is controllable

Example

Consider the following system (inverted pendulum, example 2.8 in Chen's book)

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}}_{B.} u$$



$$y = [1 \quad 0 \quad 0 \quad 0]\, x$$

$$C = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix} \\ \begin{array}{cccc} B & AB & A^2B & A^3B \end{array} \end{array}$$

$\underline{\text{rank}(C) = 4.}$

(A, B) is controllable $\Leftrightarrow$ $W_c(t)$ is nonsingular $\forall t \geq 0$.

$$W_c(t) = \int_0^t \underline{e^{A\tau} B B^T e^{A^T\tau}} \, d\tau.$$

"$\Leftarrow$" $\quad e^{A\tau} B B^T e^{A^T\tau} = \| B^T e^{A^T\tau} \|_2^2 > 0.$

$\underline{W_c^{-1}}$

if given initial state x_0, for any given x_1,

find an input such that $x(t_1) = x_1$.

$$u(t) = -B^T e^{A^T(t_1 - \tau)} W_c^{-1}(t_1)[e^{At_1} x_0 - x_1]$$

$$x(t_1) = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) \, d\tau$$

$$= e^{At_1} x_0 - \underbrace{\int_0^{t_1} e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} d\tau}_{W_c(t_1)} \cdot W_c^{-1}(t_1)[e^{At_1} x_0 - x_1]$$

$$= e^{At_1} x_0 - e^{At_1} x_0 + x_1$$

$$= x_1$$

"$\Rightarrow$" if (A, B) is controllable, then $W_c(t)$ is nonsingular for all $t \geq 0$. $\quad W_c(t) \succ 0.$

by constructing contradiction.

Assume (A, B) is ctrb, but $W_c(t_1)$ is singular.

if A is singular, A has 0 eigenvalue.

$\exists\, v \neq 0,\ Av = \lambda v = 0 \Rightarrow v^T A v = 0.$

then there exists a vector $\underline{V \neq 0}$

$$V^T W_c(t_1) V = 0.$$

$$\Rightarrow \int_0^{t_1} V^T e^{A(t_1-\tau)} B B^T e^{A^T(t_1-\tau)} V \, d\tau = 0.$$

$x^T x = \|x\|_2^2$

$$= \int_0^{t_1} \underbrace{\| B^T e^{A^T(t_1-\tau)} V \|_2^2}_{\text{always } \geq 0} \, d\tau = 0.$$

$$\Rightarrow B^T e^{A^T(t_1-\tau)} V \equiv 0 \quad \forall \tau \in [0, t_1]$$

(A, B) is ctrb, for the initial state $x(t_0) = e^{-At_1} V$,

and the final state $x(t_1) = 0$.

$$x(t_1) = 0 = e^{At_1} e^{-At_1} V + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) \, d\tau.$$

multiply by V^T on both sides.

$$0 = V^T V + \underbrace{\int_0^{t_1} \underbrace{V^T e^{A(t_1-\tau)} B}_{\equiv 0 \ \forall \tau} u(\tau) \, d\tau}_{0}.$$

$$\Rightarrow 0 = V^T V \Rightarrow \underline{V = 0}$$

$V = 0$ contradict with assumption $V \neq 0$.

then $W_c(t)$ is nonsingular.

PBH test

Popov - Belevitch - Hautus

(A, B) is controllable $\Leftrightarrow$ rank $[A-\lambda I \;\; B] = n \;\; \forall \lambda \in \mathbb{C}$

$\text{rank}(A-\lambda I) = n$ for $\lambda \notin \text{eig}(A)$.

if $\lambda \in \text{eig}(A)$ $\det(A-\lambda I) = 0$. rank $(A-\lambda I) < n$.

we only care about $\lambda \in \text{eig}(A)$

B needs to "increase rank".

$\Rightarrow$ $B \notin \text{rank}(A-\lambda_i I)$ $\forall \lambda_i$

fundamental theory of linear algebra

$\text{rank}(A-\lambda_i I) + \text{null}(A-\lambda_i I)^T = n$.

v_i. is left eigenvector of A.

B needs to have some component in each left eigen vector direction. if the columns of B orthogonal to a left eigenvect of A, B cannot have component in this direction

* there is no left eigenvector of A orthogonal to the columns of B.
* (Advanced topic) if B is a random vector then (A, B) will be ctrb with high probability.

$(\text{rank}[A-\lambda I \;\; B] = n)$

* (A, B) is ctrb $\Leftrightarrow$ there is no eigenvector of A^T in the null space of B^T

$v^T B = 0.$

$\Leftrightarrow$ there is no left eigenvector of A orthogonal to columns of B

$\Leftrightarrow$ rank $[A - \lambda_i I \quad B]$ for $\lambda \in eig(A)$.

By the fundamental theorem of linear algebra,

Codomain $(A) = R(A) \oplus N(A^T)$

$\dim N([A-\lambda I \quad B]^T) + \text{rank}[A-\lambda I \quad B] = n.$

if rank $[A - \lambda I \quad B] = n.$

then $N\left(\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix}\right) = \{0\}$

$\Rightarrow$ there is no $v \neq 0$ s.t. $A^T v = \lambda v$ & $B^T v = 0.$

"$\Rightarrow$" if (A, B) is ctrb, then there is NO eigenvector of A^T lives in $N(B^T)$.

Assume (A, B) is ctrb there exists an eigenvalue $A^T v = \lambda v$ $(v \neq 0)$ and $B^T v = 0$.

$C = [B \ \ AB \ \ \cdots \ \ A^{n-1}B]$

$$C^T = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} \qquad C^T v = \begin{bmatrix} B^T v = 0 \\ B^T A^T v \ \lambda v \to \lambda B^T v \\ \vdots \\ B^T (A^T)^{n-1} v \end{bmatrix} = 0$$

$v \neq 0$.

$\text{rank}(C^T) = \text{rank}(C) < n$.

contradicts with (A, B) is ctrb.

"$\Leftarrow$" contrapositive. $P \to Q$. $\Updownarrow$ $\neg Q \to \neg P$. inverse $\neg P \to \neg Q$. converse $Q \to P$

if there is NO eigenvector of A^T in $N(B^T)$, then (A, B) is ctrb.

☆ if (A, B) is not ctrb, then there exists eigenvector of $A^T \in N(B^T)$

$$\bar{A} = T^{-1} A T = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = T^{-1} B = \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix}$$

$A = T \bar{A} T^{-1}$

$\bar{A}_{22} W_{22} = \lambda W_{22}$

$$w = (T^{-1})^T \begin{bmatrix} 0 \\ w_{22} \end{bmatrix}$$

$$W^T \underline{A} = \begin{bmatrix} 0 & W_{22}^T \end{bmatrix} \cancel{T^{-1}} \cancel{T} \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} 0 & W_{22}^T \end{bmatrix} \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} T^{-1}$$

$$= \begin{bmatrix} 0 & \underline{W_{22}^T \bar{A}_{22}} \end{bmatrix} T^{-1}$$

$$= \lambda \underline{\begin{bmatrix} 0 & W_{22}^T \end{bmatrix} T^{-1}} = \lambda \underline{W^T}$$

$$W^T B = \begin{bmatrix} 0 \\ W_{22} \end{bmatrix}^T T^{-1} T \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & W_{22}^T \end{bmatrix} \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix} = 0$$

This proof needs information about controllable canonical form.

Example

$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

1. $C = [B \ \ AB] = \begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix}$ $\quad$ rank (C) = 2.

$A^2B \ \ A^3B.$

2. $t > 0$. $t_1 = 2$. $\quad W_c(2) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$

$$= \int_0^2 \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} [0.5 \ \ 1] \begin{bmatrix} e^{-0.5\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$\text{eig}(W_c) = 0.0083$
$\qquad 0.6987.$

3. PBH test. $[A - \lambda I \ \ B]_{n \times (n+p)}$ $\quad B = aV_1 + bV_2$

$\text{eig}(A) = \underset{V_1}{-0.5}, \underset{V_2}{-1}$ $\quad V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad = 0.5 V_1 + 1 \cdot V_2$

$[A - (-0.5) I \ \ B] = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & -0.5 & 1 \end{bmatrix}$ $\leftarrow$ rank = 2.

$[A - (-1) I \ \ B] = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ rank = 2.

4. $\lambda(A) < 0$. $\quad$ for $AW + WA = -BB^T$

Lyapunov equation

Lyap (A, Q) in Matlab $\quad Q = BB^T$

$$W = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

$\text{eig}(W) = 0.019$
$\qquad 0.731.$

Test if a system is controllable

Theorem (6.1 in Chen's book)

1. The n-dimensional pair (A,B) is controllable.

2. The $n \times n$ matrix

$$W_c(t) = \int_0^t e^{A\tau}BB'e^{A'\tau}d\tau = \int_0^t e^{A(t-\tau)}BB'e^{A'(t-\tau)}d\tau$$

   is nonsingular for any $t > 0$

3. The $n \times np$ controllability matrix to test if a system is ctrb.

   compute $C = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$ then compute rank(C)

   has rank n (full row rank)

4. The $n \times (n+p)$ matrix $[A - \lambda I \quad B]$ has full row rank at every eigenvalue, λ, of A

5. If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$AW_c + W_cA' = -BB'$$

   is positive definite. The solution is called the Controllability Grammian and can be expressed as

$$W_c = \int_0^\infty e^{A\tau}BB'e^{A'\tau}d\tau$$

Controllability indices

We consider the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} x + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B.} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

$n=4.$

$p=1. \quad C \in \mathbb{R}^{n \times n.}$

$\mathrm{rank}(C_1) = 4.$

$A^2 b_1 = a_1 b_1 + a_2 A b_1.$

$$C = [B \quad AB \quad A^2B \quad A^3B]$$

C_1

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -4 \\ 1 & 1 & -2 & 0 & 0 & -4 & 2 & 0 \end{bmatrix}$$

$b_1 \quad Ab_1$

Controllability indices

$A^4 b_1$. $A^5 b_1$

Assume that B has rank p (full column rank).

The controllability matrix can be written as $A^3 b_1$ depends on

$$C = \left[b_1 \ \cdots \ b_p \ \vdots \ Ab_1 \ \cdots \ Ab_p \ \vdots \ A^2 b_1 \cdots \vdots \ A^{n-1}b_1 \ \cdots \ A^{n-1}b_p \right]$$

if $A^i b_m$ depends on $\{b_m, Ab_m, \cdots A^{i-1} b_m\}$.

b_1

The linearly independent columns associated with b_m: so does $A^{i+1} b$

$\{b_m, Ab_m, \cdots A^{u_m - 1} b_m\}$

The controllability indices

$\{u_1, u_2, \cdots u_p\}$.

$u = \max\{u_1, u_2, \cdots u_p\}$ controllability index