$$y(k) = Cx(k) = C A^k x(0) + \sum_{j=0}^{k-1} \underbrace{CA^{k-1-j}Bu(j)}_{CA^{k-1-j}Bu(j)}$$

$Kx(k-1)$

$$x(k) = \left(e^{(A+BK)\Delta t}\right)^k x(0)$$

$\bar{A}$

$$y(k) = Cx(k) = C\left(e^{(A+BK)\Delta t}\right)^k x(0)$$

$$\begin{aligned} y(0\Delta t) &= Cx(0) + \boxed{v(0)} \\ y(1\Delta t) &= C\bar{A}x(0) + v(1) \\ y(2\Delta t) &= C\bar{A}^2 x(0) \\ y(3\Delta t) & \\ &\vdots \\ y(100) &= C\bar{A}^{100} x(0) \end{aligned} \longrightarrow \begin{bmatrix} C \\ C\bar{A} \\ C\bar{A}^2 \\ \vdots \\ C\bar{A}^{100} \end{bmatrix} x(0)$$

101×3

M

101×1

$$\boxed{Y} = Mx(0) + \underbrace{\begin{bmatrix} v(0) \\ \vdots \\ v(101) \end{bmatrix}}_{V}$$

$\mathcal{N}(0,1)$

$$\boxed{Y+V} = Mx(0) \Rightarrow (M^T M)^{-1} M^T (Y-V) = x(0)$$

Converting continuous time to discrete time simulation:

direction of motion

next location

$$\dot{x} = Ax + Bu \longrightarrow x^+ = \bar{A}x + \bar{B}u$$

integrate

$$x(\Delta t + t) = e^{A\Delta t} x(t) + \int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)} B\, u(\tau)\, d\tau$$

$\tau = \tau' + t$

$\tau' \in [0, \Delta t]$

$$x(\Delta t + t) = e^{A\Delta t} x(t) + \int_0^{\Delta t} e^{A(\Delta t - \tau')} B\, u(\tau'+t)\, d\tau'$$

zero order hold.

$$x^+ = \bar{A}\, x(t) + \underbrace{\int_0^{\Delta t} e^{A(\Delta t-\tau')} B\, d\tau'}_{\bar{B}}\, u(t)$$

$u(\tau'+t) = u(t)$ for $\tau' \in [0, \Delta t]$

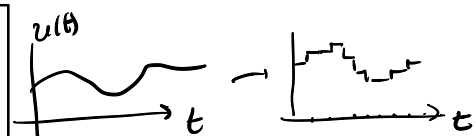
$$\bar{B} = A^{-1}(e^{A\Delta t} - I)B$$

$$x(k+1) = \bar{A}x(k) + \bar{B}u(k)$$

first 2 terms Taylor

$$\boxed{\begin{aligned} \bar{A} &= e^{A\Delta t} \approx (I + \Delta t A) \\ \bar{B} &= A^{-1}(e^{A\Delta t} - I)B \approx \Delta t B \end{aligned}}$$

first 2 terms of Taylor



Matlab:

$$\bar{A}^k = (e^{A\Delta t})^k = e^{Ak\Delta t} = e^{At}$$

Simulating

CT: ODE45 (sophisticated integration Newtons, Runga Kutta)

DT: $x(k) = A^k x(0) + [\_\_\_\_\_\_\_\_]$

FREQUENCY DOMAIN

- Laplace Transform *]
  - Fourier Transform
  - Frequency/time duality
- Transfer functions]
- Bode Plots *]
- FREQ ⟺ STATE SPACE]
- (Z - TRANSFORM)
- CIRCULANT] → most interesting]

HOMEWORK 9

$\sin(\omega t)$ t: time

ω: oscillation speed

Laplace Transforms (diff eq → algebraic eqn)

$f(t)$ — func of time

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

Intuition: inner product

$y, x \in \mathbb{R}^n$: $y^T x$ → inner product between $x, y \in \mathbb{R}^n$

$f(t), g(t)$: $f(t)$ – inf dim vector
$g(t)$ – "

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leftarrow i$ $f(t) = \begin{bmatrix} f(0) \\ \vdots \\ f(T) \end{bmatrix} \leftarrow t$

ith index ~ time t

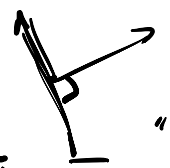
$\sum_i y_i x_i$ fin. dim

$\int_0^T f(t)g(t)\,dt \leftarrow$ inner product.

$y^T x$: "projection of x onto y"

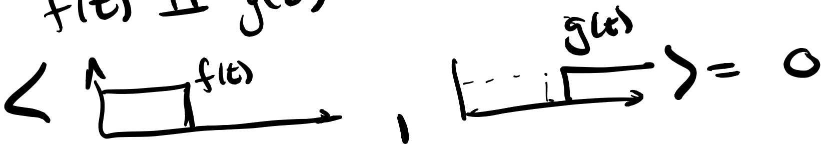
$\mathrm{proj}_y x = \frac{y}{|y|}\frac{y^T}{|y|}\boxed{x}$

$x \perp y$ $y^T x$



$\int_0^T f(t)g(t)\,dt$ = "projection of $g(t)$ onto $f(t)$"

$f(t) \perp g(t)$ $= 0$



$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

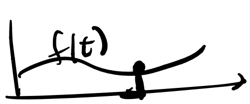
$g_s(t) = e^{-st}$ s: parameter, complex #

freq oscillation & decay expansion rate

"projecting $f(t)$ onto e^{-st}" $\Leftarrow$

$\mathcal{F}(f) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \Leftarrow$ Fourier

projection of $f(t)$ onto functions of the form



basis of $\delta(t)$

delta function

$e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$

coordinate transformation

$\downarrow \mathcal{F}(f)$

basis of oscillations

$f(t)$

func of time

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

s : complex # $\quad s = a + \omega i$

$= re^{i\theta}$

$t \in \mathbb{R}$

$s \in \mathbb{C}$

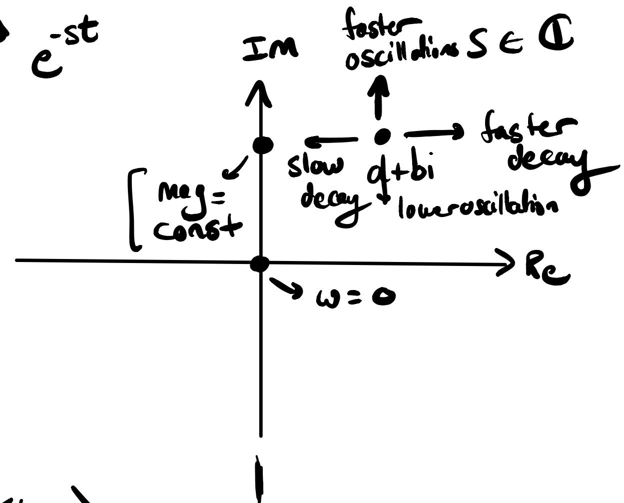
$\rightarrow e^{-st}$

$s = a + \omega i$

$$e^{-st} = e^{-at} e^{-wit}$$

mag decay rate

oscillating term mag $= 1$



if $s = \omega i$

$\mathcal{L}(f(t))$ restricted to imag. axis $= \mathcal{F}(f(t))$

Time Signal:

$f(t)$

$f(t) = \delta(t)$ impact

$f(t) = \delta(t - \tau)$

Laplace (Fourier transform)

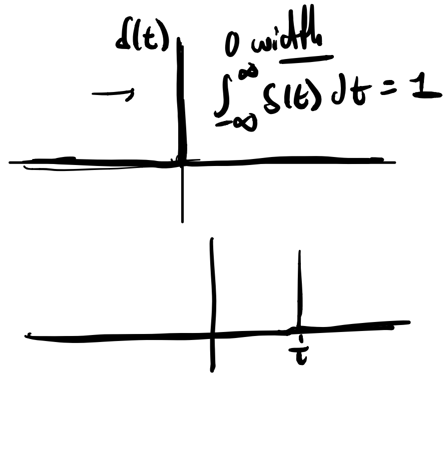
$F(s)$

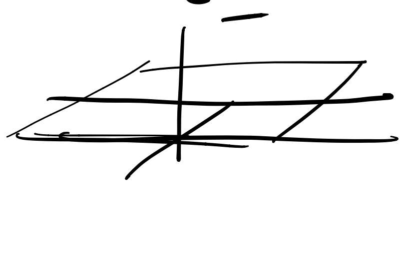
$$F(s) = \int_0^\infty e^{-st} \delta(t)\, dt = \underline{1}$$

$$= e^{-s(0)} = 1$$

$$F(s) = \int_0^\infty e^{-st} \delta(t-\tau)\, d\tau$$

$$= e^{-s\tau}$$





$f'(t) = \frac{df}{dt}$ $\qquad \mathcal{L}(f') = s\,\underbrace{F(s)}_{\mathcal{L}(f(t))} - f(0)$

components of $F(s)$
with larger values of s
impact the derivative more

Convolution

$g(t), f(t)$ $\qquad G(s), F(s)$

$$(f*g) = \int_0^t g(t-\tau) f(\tau)\, d\tau \qquad \mathcal{L}(f*g) = G(s)\,F(s)$$

Compare to impact of control

$$\int_0^t \underbrace{e^{A(t-\tau)}}_{g(t-\tau)} B \underbrace{u(\tau)}_{f(\tau)}\, d\tau$$

$f(\tau) \rightarrow$ control/input signal

$g(t) = e^{At}B$

system response

to get impact of $u(t)$ on state $x(t)$

$\Rightarrow$ convolve $u(t)$ w/ sys response $e^{At}B$

step function

$u(t)$  $\longrightarrow \mathcal{L}(u(t)) = \frac{1}{s}$

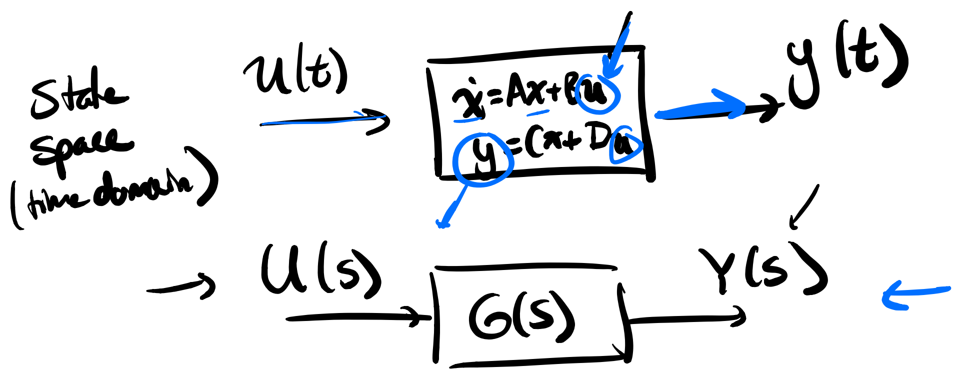
$u(t-\tau)$ $\qquad \mathcal{L}(u(t-\tau)) = \frac{1}{s} e^{-\tau s}$

$f(t-\tau)\,u(t-\tau) \longrightarrow \mathcal{L} = e^{-\tau s} F(s)$

"waiting till $t=\tau$ to apply f"

shift in time $\leftarrow$

Freq Domain System Modeling



convolution in time dom $G(s)$: system response
transfer function

$$Y(s) = \underbrace{G(s)}_{\text{sys response}} \cdot \underbrace{U(s)}_{\text{control input}}$$

Transfer Function : $G(s)$ "how are oscillating signals propagated through the system."

$G(j\omega)$ = complex #

$G(j\omega)$: tells how a signal of the form $u(t) = \sin(\omega t)$ propagates through the sys.

input oscillating at rate ω (amplitude is const.)

$r = |G(j\omega)|$: tells us how much signals w/ freq. ω get amplified

$\phi = \angle G(j\omega)$: phase lag of output

$u(t)$ steady state behavior

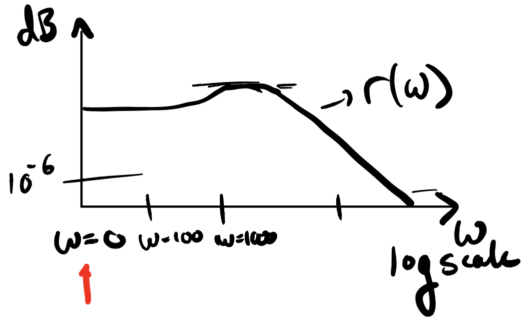
$$r(\omega)\sin(\omega t + \phi) = G(j\omega)\sin(\omega t)$$

rad/s

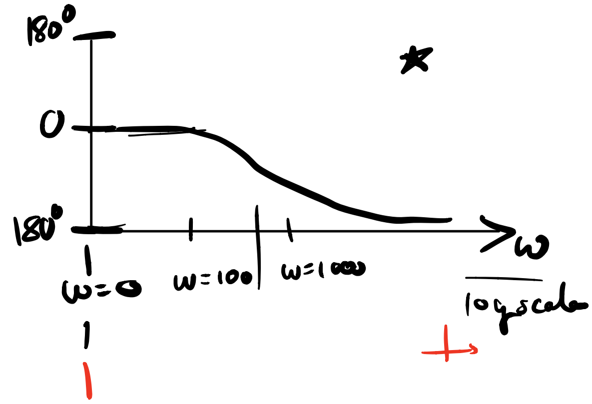
Hertz ($\times 2\pi$)

Bode Plots

$|G(j\omega)| = r$



$\angle G(j\omega) = \phi$



State Space

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Longrightarrow Y(s) = G(s)U(s)$$

$$\mathcal{L}(\dot{x}) = \mathcal{L}(Ax + Bu)$$

$$s\underbrace{\mathcal{L}(x)}_{X(s)} - x(0) = A\underbrace{\mathcal{L}(x)}_{X(s)} + B\underbrace{\mathcal{L}(u)}_{U(s)}$$

$$sX(s) - x(0) = AX(s) + Bu(s)$$

$$\rightarrow X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) \Leftarrow$$

compare: $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

$$\boxed{\mathcal{L}(e^{At}) = (sI - A)^{-1}}$$

$\left(e^{At} * Bu(t)\right)$

$(sI-A)^{-1} \cdot BU(s)$

plugging in

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{\text{transient piece}} + \left(\underbrace{C(sI-A)^{-1}B + D}_{\text{steady state sys response}}\right)\underbrace{u(s)}_{\text{control input}}$$

init cond.

steady state

$$Y(s) = G(s)u(s) \quad \boxed{G(s) = C(sI-A)^{-1}B + D}$$

memorize

Note: $G(s)$ invariant under coord transforms.

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \to \quad x = Pz$$

$$P\dot{z} = APz + Bu$$

$$\dot{z} = P^{-1}APz + P^{-1}Bu$$

$$y = CPz + Du$$

$$G(s) = CP\left(sI - P^{-1}AP\right)^{-1}P^{-1}B$$

$$\to CP\left(sP^{-1}P - P^{-1}AP\right)^{-1}P^{-1}B$$

$$CP\left(P^{-1}(sI-A)P\right)^{-1}P^{-1}B$$

$$CPP^{-1}(sI-A)^{-1}PP^{-1}B$$

$$= C(sI-A)^{-1}B$$

$$G(s) = C(sI-A)^{-1}B + D$$

$$Y(s) = \_\_\_\_ + D\,u(s)$$

DIAG. A ...

$$A = PDP^{-1}$$

$$P = \underbrace{\begin{bmatrix} p_1 \cdots p_n \end{bmatrix}}_{\text{right evecs}} \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad P^{-1} = \underbrace{\begin{bmatrix} -\xi_1^T- \\ \vdots \\ -\xi_n^T- \end{bmatrix}}_{\text{left evecs}}$$

$$= C(s\underbrace{I}_{PP^{-1}} - PDP^{-1})^{-1}B + D$$

$$= CP(sI-D)^{-1}P^{-1}B + D$$

$$C\begin{bmatrix} p_1 \cdots p_n \end{bmatrix} \begin{bmatrix} s-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & s-\lambda_n \end{bmatrix}^{-1} \begin{bmatrix} \xi_1^T \\ \vdots \\ \xi_n^T \end{bmatrix} B$$

input directions left evecs

output directi. right evecs

$$\begin{bmatrix} Cp_1 \cdots Cp_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} \xi_1^T B \\ \vdots \\ \xi_n^T B \end{bmatrix}$$

$$G(s) = \sum_i Cp_i\,\xi_i^T B \,\frac{1}{s-\lambda_i} + D \qquad s = \lambda_i$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{bmatrix} p_1 \end{bmatrix} \begin{bmatrix} \xi_1^T \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

$$\begin{bmatrix} T \end{bmatrix}$$

$$= C\,\frac{\mathrm{Adj}(sI-A)}{\det(sI-A)}\,B + D$$

$$(sI-A)^{-1}$$

if $C \in \mathbb{R}^{1\times n}$ $B \in \mathbb{R}^{n\times 1}$ $M^{-1} = \frac{\mathrm{Adj}(M)}{\det(M)}$

single input / single output system (SISO)

$$G(s) = \frac{C\,\mathrm{Adj}(sI-A)B + D\det(sI-A)}{\det(sI-A)} = \frac{N(s)}{\chi_A(s)}$$

characteristic poly of A

$\chi_A(s)$: degree n.

$N(s)$: deg. $m \le n$
if $D=0$, $m<n$ $\Big\} \to$ causal systems (proper)

$$G(s) = \frac{N(s)}{\chi_A(s)} = \frac{(s-z_1)\cdots(s-z_m)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

$\lambda_1 \cdots \lambda_n$: eigenvalues, poles of $G(s)$ $\to$ blows up.

$z_1 \cdots z_n$: zeros of $G(s)$ $\} \to$ transfer func is 0

Mag: $|G(j\omega)| = r$ Phase $\angle G(j\omega) = \phi$

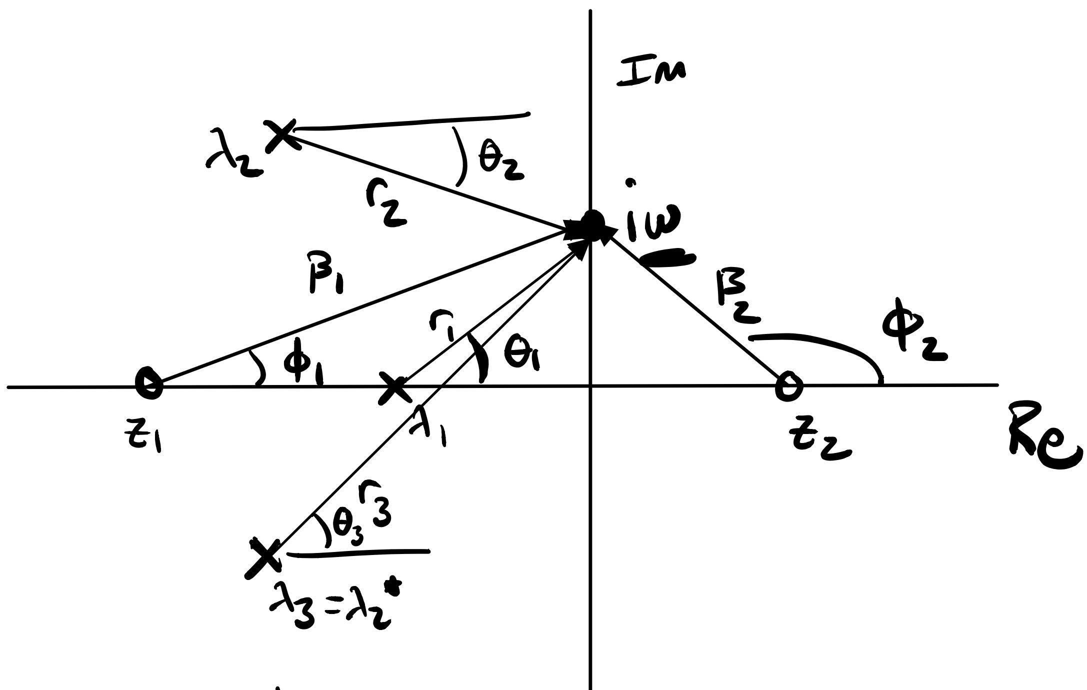
$$G(j\omega) = re^{i\phi}$$

$$G(s) = \frac{(s-z_1)\cdots(s-z_m)}{\underbrace{(s-\lambda_1)\cdots(s-\lambda_n)}_{\text{Complex}}} = \frac{\prod_j \beta_j e^{i(\phi_1+\cdots+\phi_m)}}{\prod_k r_k e^{i(\theta_1+\cdots+\theta_n)}}$$

$$i\omega-\lambda_k = r_k e^{i\theta_k}$$

$$G(s) = \frac{\prod_j \beta_j}{\prod_k r_k} e^{i\left(\sum_j \phi_j - \sum_k \theta_k\right)} \qquad i\omega - z_j = \beta_j e^{i\phi_j}$$

$$|G(j\omega)| = \frac{\prod_j \beta_j}{\prod_k r_k} \qquad \angle G(j\omega) = \sum_j \phi_j - \sum_k \theta_k$$

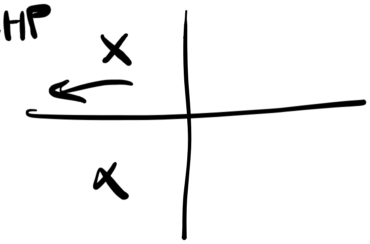


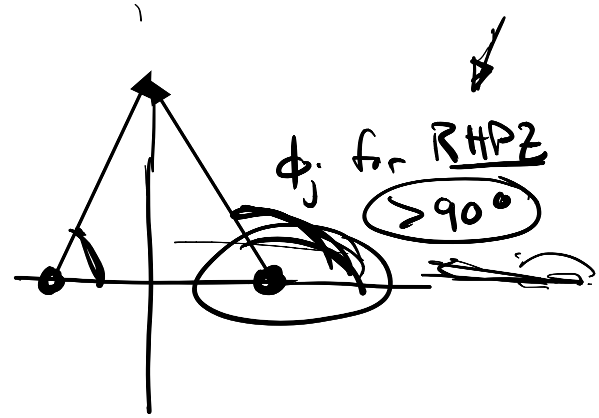
$$\frac{|i\omega - z_j|}{|i\omega-\lambda_1||i\omega-\lambda_2||i\omega-\lambda_3|} = |G|$$

faster decaying λ_k: reduce $|G(i\omega)|$

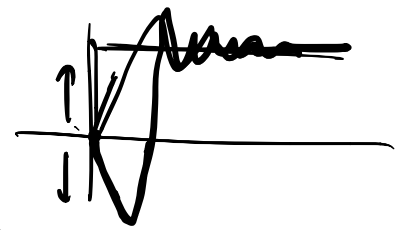
Only talk about transfer functions for stable

OLHP





Transfer Function:



stable, minimum phase → well behaved

↓ no RHP poles

↓ no RHP zeros

Time Delay:

$G(s)$ add time delay of τ → $e^{-\tau s} G(s)$

$\downarrow$

$- e^{-i\omega\tau} G(i\omega)$

$\swarrow$

$r e^{i(\phi - \underline{\omega\tau})}$

RHP zeroes & time delays have same effect.

Multi input / Multi output (MIMO)

$$G(s) = C(sI - A)^{-1}B + D = \frac{C\,\mathrm{Adj}(sI - A)B}{\det(sI - A)}$$

$$G(s) = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \rightarrow$$

$$|G(s)| \rightarrow$$ singular values of G