AE 513 - Multivariable Control - Autumn 2019

Homework 1

Due Date: Thursday, Oct 10^{th}, 2019 at 11:59pm

1. **Inner Products**

   (a) **(PTS: 0-2)** Prove $y^\intercal x = |x||y|\cos\theta$ using the definition of the 2-norm and the law of cosines.

   (b) **(PTS: 0-2)** Prove the parallelogram law:

   $$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$$

2. **Projections**

   (a) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^\intercal$ onto $y = [1, 1, -2]^\intercal$.

   (b) **(PTS: 0-2)** Compute the projection of $x = [1, 2, 3]^\intercal$ onto the range of

   $$Y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. **Block Matrix Computations**

   Multiply the following block matrices together. In each case give the required dimensions of the sub-blocks of B. If the dimensions are not determined by the shapes of A, then pick a dimension that works.

   (a) **(PTS: 0-2)**

   $$A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1K} \\ \vdots & & \vdots \\ B_{N1} & \cdots & B_{NK} \end{bmatrix}, \quad AB = ? \tag{1}$$

   where $A_{11} \in R^{m_1 \times n_1}$, $A_{1N} \in R^{m_1 \times n_N}$, $A_{M1} \in R^{m_M \times n_1}$, and $A_{MN} \in R^{m_M \times n_N}$.

   (b) **(PTS: 0-2)**

   $$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \quad AB = ? \tag{2}$$

   where $A_1 \in R^{1 \times n}$ and $A_m \in R^{1 \times n}$.

(c) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad AB = ? \tag{3}$$

where $A_1 \in R^{m \times 1}$ and $A_n \in R^{m \times 1}$.

(d) **(PTS: 0-2)**

$$A = \begin{bmatrix} - & A_1 & - \\ \vdots & & \vdots \\ - & A_m & - \end{bmatrix}, \quad D \in R^{n \times n}, \quad B = \begin{bmatrix} | & \cdots & | \\ B_1 & & B_k \\ | & \cdots & | \end{bmatrix}, \qquad ADB = ? \tag{4}$$

where $A_1 \in R^{1 \times n}$, $A_m \in R^{1 \times n}$.

(e) **(PTS: 0-2)**

$$\begin{bmatrix} | & \cdots & | \\ A_1 & & A_n \\ | & \cdots & | \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} - & B_1 & - \\ \vdots & & \vdots \\ - & B_n & - \end{bmatrix}, \qquad ADB = ? \tag{5}$$

where $A_1 \in R^{m \times 1}$, $A_n \in R^{m \times 1}$, $d_{ij} \in R$.

(f) **(PTS: 0-2)**

$$A \in R^{m \times n}, \quad \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}, \qquad AB = ? \tag{6}$$

(g) **(PTS: 0-2)**

$$A = \begin{bmatrix} -A_1- \\ \vdots \\ -A_m- \end{bmatrix}, \quad B, \qquad AB = ? \tag{7}$$

where $A_1, A_m \in R^{1 \times n}$.

4. **Traces and Determinants**

(a) **(PTS: 0-2)**: The determinant of a diagonal matrix is the product of the diagonal elements. Draw (or describe) a picture illustrating this fact based on the determinant being the signed volume of the transformed unit cube.

Assume that $A \in R^{n \times n}$ is diagonalizable and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Use the properties of traces and determinants and the fact from part (a) to show that

(b) **(PTS: 0-2)**: $\mathrm{Tr}(A) = \sum_i \lambda_i$

(c) **(PTS: 0-2)**: $\det(A) = \prod_i \lambda_i$

5. **Bases**

   (PTS: 0-2) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \left\{ \left[x_1, x_2, \ldots, x_5 \right]^\intercal \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}$$

   Find a basis for U.

6. **Matrix Change of Basis**

   Let $\varepsilon = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. Consider another basis for R^2 given by the columns of V.

$$V = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   (a) **(PTS: 0-2)** Let $x = [1, -1]^T$. Find the coordinates of x with respect to the basis V, ie. find z such that $x = Vz$

   (b) **(PTS: 0-2)** Let $\theta = \pi/2$ and

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   Consider A acting on x, ie. $x' = Ax$. Let $x = Vz$ and $x' = Vz'$. Find B such that $z' = Bz$.

7. **Range- Nullspace** Let $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the range and nullspace of A (and similarly for $\mathcal{R}(A^T)$ and $\mathcal{N}(A^T)$.

   (a) **(PTS: 0-2)** Suppose $y \in \mathcal{R}(A)$ and $x \in \mathcal{N}(A^T)$. Show that $x \perp y$, ie. $x^T y = 0$.

   (b) **(PTS: 0-2)** Consider $A \in R^{5\times 10}$. Suppose A has only 3 linearly independent columns (the other 7 are linearly dependent on the first 3). What is the dimension of $\mathcal{R}(A)$? What is the dimension of $\mathcal{N}(A^T)$?

   (c) **(PTS: 0-2)** What is the dimension of $\mathcal{N}(A)$? What is the dimension of $\mathcal{R}(A^T)$?

8. **Similarity Transforms and Diagonalization**

   Suppose $p_1, p_2 \in R^2$ are linearly independent right eigenvectors of $A \in R^{2\times 2}$ with eigenvalues $\lambda_1, \lambda_2 \in R$ such that $\lambda_1 \neq \lambda_2$. Suppose that

$$p_1^T p_2 = 0, \qquad |p_1| = 1, \qquad |p_2| = 2$$

   (a) **(PTS: 0-2)** Write an expression for a 2×2 matrix whose rows are the left-eigenvectors of A

   (b) **(PTS: 0-2)** Write an expression for a similarity transform that transforms A into a diagonal matrix.

9. **Spectral Mapping Theorem**

   Consider a diagonalizable matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$ and a polynomial function $f : R^{n\times n} \to R^{n\times n}$.

   (a) **(PTS: 0-2)** Show that the eigenvectors (left and right) of $f(A)$ are the same as the eigenvectors of A.

(b) **(PTS: 0-2)** Show that the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

10. Find a dynamical system of interest to you (or from the list in the announcement) and write out the dynamics. Nonlinear dynamics are welcome. Extra credit if you type them up and include a good diagram.