AE 513 - Multivariable Control - Autumn 2019

Homework 2

Due Date: Thursday, Oct 17^{th}, 2019 at 11:59pm

1. **Similar Eigenvalues**

   (a) **(PTS: 0-2)** Let $A \in \mathbb{R}^{n\times n}$ and let $T \in \mathbb{R}^{n\times n}$ be any non-singular matrix. Show that the eigenvalues of A are the same as those of $T^{-1}AT$.

   (b) **(PTS: 0-2)** Let $A, B \in \mathbb{R}^{n\times n}$ be invertible matrices. Show that the eigenvalues of AB are the same as those of BA.

2. **Computing Eigenvalues and Diagonalization**

   Compute eigenvalues and right eigenvectors for each of the following matrices. Write out a diagonalization for each matrix. If the matrix has complex eigenvalues, then write it in both of these forms.

$$\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}(u - vi) & \frac{1}{\sqrt{2}}(u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}}(w^T + y^T i)- \\ -\frac{1}{\sqrt{2}}(w^T - y^T i)- \end{bmatrix} = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} -w^T- \\ -y^T- \end{bmatrix}$$

   (a) **(PTS: 0-2)** Eigenvalues, **(PTS: 0-2)** Eigenvectors, **(PTS: 0-2)** Diagonal form, **(PTS: 0-2)**, Complex form?

$$A = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

   (b) **(PTS: 0-2)** Eigenvalues, **(PTS: 0-2)** Eigenvectors, **(PTS: 0-2)** Diagonal form, **(PTS: 0-2)**, Complex form?

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

   (c) **(PTS: 0-2)** Eigenvalues, **(PTS: 0-2)** Eigenvectors, **(PTS: 0-2)** Diagonal form, **(PTS: 0-2)**, Complex form?

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

   (d) **(PTS: 0-2)** Eigenvalues, **(PTS: 0-2)** Eigenvectors, **(PTS: 0-2)** Diagonal form, **(PTS: 0-2)**, Complex form?

$$A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

3. **Cayley-Hamilton Theorem**

   (a) **(PTS: 0-2)** The eigenvalues of a matrix A are roots of its characteristic polynomial, $\chi(\lambda) = \det(\lambda I - A)$, ie. $\det(\lambda_i I - A) = 0$ if λ_i is an eigenvalue of A. Show that $\chi(A) = \mathbf{0}$ (where $\mathbf{0}$ is a matrix of zeros). (Hint: use the spectral mapping theorem).

   (b) **(PTS: 0-2)** . Suppose that $\chi(\lambda) = \det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda - 1$. Use Cayley-Hamilton to write an expression for A^6 in terms of A^2, A, I. Note that when you plug the matrix A into $\chi(\cdot)$ you replace each constant with that constant times the identity matrix, ie. $\chi(A) = A^3 - 2A^2 + A - I$.

4. **Rotation Matrices and Complex Eigenvectors**

   Consider the two rotation matrices

$$R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

   (a) **(PTS: 0-2)** Show that R_1 and R_2 commute, ie. $R_1 R_2 = R_2 R_1$ (Note that most matrices do not commute. 2×2 rotation matrices are an exception.)

   (b) **(PTS: 0-2)** Compute the inverse of R_1.

   (c) **(PTS: 0-2)** Give a physical interpretation of $R_1 R_2$ and R_1^{-1} related to the angles θ_1 and θ_2.

   (d) **(PTS: 0-2)** Consider a 2×2 real matrix A that can be diagonalized as

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \begin{bmatrix} | & | \\ (u - vi) & (u + vi) \\ | & | \end{bmatrix}^{-1} \sqrt{2}$$

   where $r \in R_+$ and $u, v \in R^2$. Show that another valid diagonalization for A is

$$A = \tfrac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix} \begin{bmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{bmatrix} \begin{bmatrix} | & | \\ (u' - v'i) & (u' + v'i) \\ | & | \end{bmatrix}^{-1} \sqrt{2}$$

   where $u' = \cos(\phi)u + \sin(\phi)v$ and $v' = -\sin(\phi)u + \cos(\phi)v$ for any angle ϕ.

5. **Vector Fields and Stability**

   For each of the A matrices in Question 2, consider the system differential equation

$$\dot{x} = Ax$$

   (PTS: 0-2) What are the eigenvalues of e^{At}? **(PTS: 0-2)** Decide if the system is stable. **(PTS: 0-2)** Sketch the vector field for each system labeling the eigenvectors and show a sample trajectory.

6. **Dynamical Systems**

   (a) **(PTS: 0-2)** If

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

   Show that

$$\dot{x} = \tfrac{d}{dt}x(t) = Ax(t) + Bu(t)$$

   (Hint: Leibniz integral rule will be helpful.)

(b) **(PTS: 0-2)** Consider the discrete time (time varying) update equation

$$x[t+1] = A[t]x[t] + B[t]u[t]$$

Write an expression for $x[t]$ in terms of the initial state $x[0]$, $u[0], ..., u[t-1]$, $A[0], \ldots, A[t-1]$, and $B[0], \ldots, B[t-1]$.