AE 513 - Multivariable Control - Autumn 2019

Homework 3

Due Date: Thursday, Oct 24^{th}, 2019 at 11:59pm

1. **Symmetric Matrices and Hessians**

   Let $Q \in R^{n\times n}$ be symmetric, $Q = Q^\intercal$, with distinct eigenvalues $\lambda_i \neq \lambda_j$.

   (a) **(PTS: 0-2)** Show the eigenvectors are orthogonal.
   Hint: for v_i, v_j look at quantity $v_i^\intercal Q v_j$.

   (b) **(PTS: 0-2)** Show that if $x^\intercal Q x > 0$ for all $x \in R^n$, then $\lambda_i > 0$ for all i.

   (c) **(PTS: 0-2)** Show that if $\lambda_i > 0$ for each eigenvalue, then $x^\intercal Q x > 0$ for all $x \in R^n$

   (d) **(PTS: 0-2)** Assume A is not symmetric. Let $f(x) = x^\intercal A x$, show that $\frac{\partial^2 f}{\partial x^2}$ is symmetric.

   (e) **(PTS: 0-2)** Assume function $f(x)$ and $f : \mathbb{R}^n \to \mathbb{R}$, show that $\frac{\partial^2 f}{\partial x^2} = \left[\frac{\partial^2 f}{\partial x^2}\right]^\intercal$

2. **Least Squares and Minimum Norm Solutions**

   (a) **(PTS: 0-2)** Consider $A \in R^{m\times n}$ where $m > n$ (A is "tall") and A has full-column rank (the columns are linear independent). Show that the least squares solution $x = (A^\intercal A)^{-1} A^\intercal y$, minimizes $|y - Ax|^2$, ie. makes Ax as close as possible to y.

   (b) **(PTS: 0-2)** Consider $A \in R^{m\times n}$ where $m < n$ (A is "fat") and A has full-row rank (the rows are linear independent). Let $x = A^\intercal (AA^\intercal)^{-1} y$ and $z \in R^n$ be any vector such that $y = Az$. Show that $|x| \leq |z|$.

3. **Controllability/Observability and Eigenvectors**

   Consider the dynamical system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

   where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$ and A is diagonalizable with right and left eigenvectors the columns and rows of P and Q respectively

$$P = \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix}, \qquad Q = P^{-1} = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_n^T & - \end{bmatrix} \tag{1}$$

   (a) **(PTS: 0-2)** Suppose there exists a left eigenvector of A, $q_i^T \in R^{1\times n}$ such that $q_i^T B = 0$. Show that the system is not controllable.

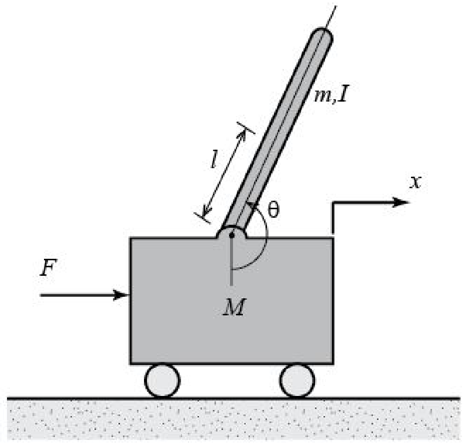
   (b) **(PTS: 0-2)** Now suppose $q_i^T B \neq 0$ for all i, but the first two eigenvalues are the same. Show that the system is not controllable.

(c) **(PTS: 0-2)** Suppose there exists a right eigenvector of A, $p \in R^n$ such that $Cp = 0$. Show that the system is not observable.

(d) **(PTS: 0-2)** Now suppose that $Cp_i \neq 0$ but the first two eigenvalues of the system are the same. Show that the system is not observable.

4. **Linearizing a System**

Given the inverted pendulum model below.



$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = F$$
$$(I+ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta$$

Linearize the equation of motion around the equilibrium point at

$$\begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where the assumptions are given as follow:

$$\theta = \pi + \phi$$
$$\dot{\theta}^2 = \dot{\phi}^2 = 0$$

small angle approximation for ϕ

5. **Controllability/Observability of Physical Systems**

For each of the physical systems given, assess each of the following questions.

(a) **(PTS: 0-2)** Determine whether or not the system is stable or unstable.

(b) **(PTS: 0-2)** Compute the eigenvectors of the system.

(c) **(PTS: 0-2)** Give a qualitative physical interpretation of each of the eigenvectors in terms of the system states, which physical states are couple together and how.

(d) **(PTS: 0-2)** Is the system controllable or not?

(e) **(PTS: 0-2)** Is the system observable or not?

(f) **(PTS: 0-2)** From the continuous time dynamics given, compute the discrete time system matrices $\bar{A}$, $\bar{B}$ for time step of $\Delta t = 0.01$ seconds.

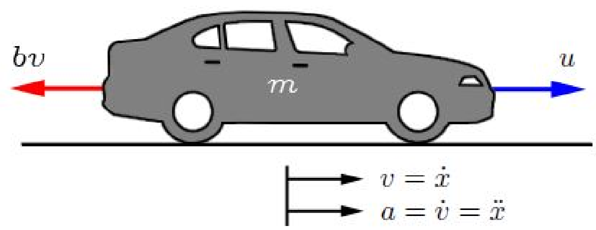
$$\bar{A} = e^{A\Delta t}, \qquad \bar{B} = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \; d\tau$$

(g) **(PTS: 0-2)** If the system is controllable, compute the minimum norm open loop controller to drive the system to 0 in 200 time steps (using the discrete time system) for the initial state given .

(h) **(PTS: 0-2)** (If the system is controllable) Plot the minimum norm control signal over time and the corresponding state trajectories.

i.

Cruise Control



ii.

System Parameters

A. $m =$ vehicle mass 1000 kg

B. $b =$ damping 50 N.s/m

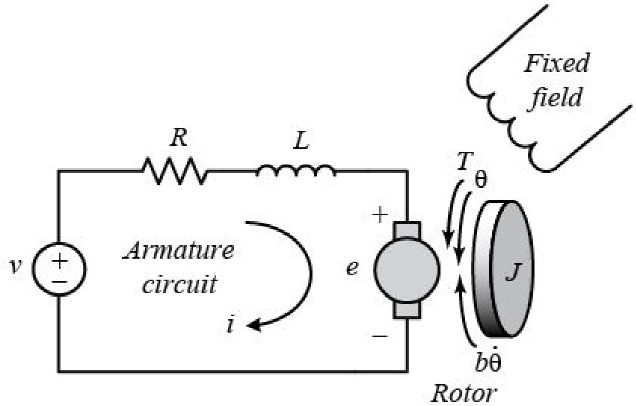
Equations of Motion:

$$m\dot{v} + bv = u$$
$$y = v$$

State-space:

$$[\dot{v}] = \left[\frac{-b}{m}\right][v] + \left[\frac{1}{m}\right][u]$$
$$y = [1][v]$$
$$[\dot{v}] = \begin{bmatrix}-0.05\end{bmatrix}\begin{bmatrix}v\end{bmatrix} + \begin{bmatrix}0.001\end{bmatrix}\begin{bmatrix}u\end{bmatrix}$$
$$y = \begin{bmatrix}1\end{bmatrix}\begin{bmatrix}v\end{bmatrix}$$
$$x[0] = 10$$

iii. DC Motor Position



System Parameters

A. J = moment of inertia of the rotor $3.2284e^{-6}$ $[kg \cdot m^2\ ]$

B. b = motor viscous friction constant $3.5077e^{-6}$ $[N \cdot m \cdot s]$

C. K_e = electromotive force constant 0.0274 $[V/rad/s]$

D. K_t = motor torque constant 0.0274 $[N \cdot m/Amp]$

E. R = electric resistance 4.0 $[Ohm]$

F. L = electric inductance $2.75e^{-6}$ [H]

Equations of Motion:

$$
\begin{aligned}
&T = K_t i \\
&e = K_e \dot{\theta} \\
&J\ddot{\theta} + b\dot{\theta} = Ki \\
&L\frac{di}{dt} + Ri = V - K\dot{\theta}
\end{aligned}
$$

State-space:

$$
\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K}{J} \\ 0 & -\frac{K}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \frac{1}{L}\end{bmatrix} V
$$

$$
y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix}
$$

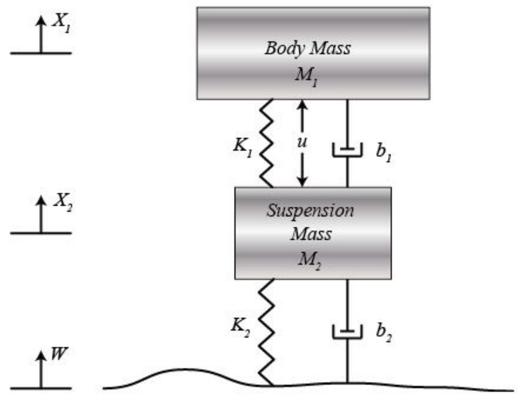
$$
\begin{bmatrix}\dot{\theta} \\ \ddot{\theta} \\ \dot{i}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 \\ 0 & -1.087 & 8487 \\ 0 & -9964 & -1.455e^6\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix} + \begin{bmatrix}0 \\ \\ 3.636e^5\end{bmatrix} V
$$

$$
y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta} \\ i\end{bmatrix}
$$

$$
x[0] = \begin{bmatrix}\frac{\pi}{2} \\ -\frac{\pi}{8} \\ 2\end{bmatrix}
$$

iv. Suspension

Model of Bus Suspension System (1/4 Bus)



System Parameters

A. M_1 = 1/4 bus body mass 2500 $[kg]$
B. M_2 = suspension mass 320 $[kg]$
C. K_1 = spring constant of suspension system 80,000 $[N/m]$
D. K_2 = spring constant of wheel and tire 500,000 $[N/m]$
E. b_1 = damping constant of suspension system 350 $[N \cdot s/m]$
F. b_2 = damping constant of wheel and tire 15,020 $[N \cdots /m]$
G. U = control force

Equations of Motion:

$$M_1\ddot{X}_1 = -b_1\left(\dot{X}_1 - \dot{X}_2\right) - K_1\left(X_1 - X_2\right) + U$$
$$M_2\ddot{X}_2 = b_1\left(\dot{X}_1 - \dot{X}_2\right) + K_1\left(X_1 - X_2\right) + b_2\left(\dot{W} - \dot{X}_2\right) + K_2\left(W - X_2\right) - U$$

State-space:

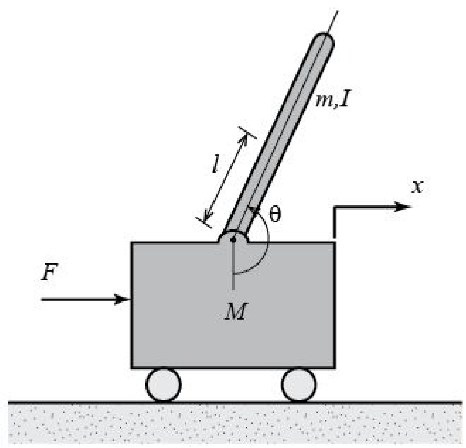
$$\begin{bmatrix} \dot{X}_1 \\ \ddot{X}_1 \\ \dot{X}_2 \\ \ddot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-b_1 b_2}{M_1 M_2} & 0 & \left[\frac{b_1}{M_1}\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) - \frac{K_1}{M_1}\right] & \frac{-b_1}{M_1} \\ \frac{b_2}{M_2} & 0 & -\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) & 1 \\ \frac{K_2}{M_2} & 0 & -\left(\frac{K_1}{M_1} + \frac{K_1}{M_2} + \frac{R_2}{M_2}\right) & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & \frac{b_1 b_2}{M_1 M_2} \\ 0 & \frac{-b_2}{M_2} \\ \left(\frac{1}{M_1} + \frac{1}{M_2}\right) & \frac{-K_2}{M_2} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$

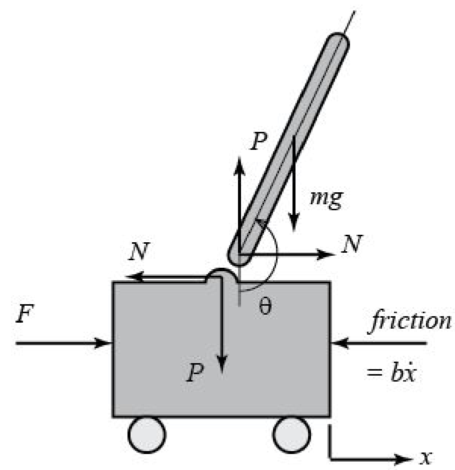
$$
\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6.571 & 0 & -25.26 & -0.14 \\ 46.94 & 0 & -48.17 & 1 \\ 1563 & 0 & -1845 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.0004 & 6.571 \\ 0 & -46.94 \\ 0.003525 & -1563 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}
$$

$$
Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}
$$

$$
x[0] = \begin{bmatrix} 0.5 \\ 2 \\ -0.5 \\ -3 \end{bmatrix}
$$

v. Inverted Pendulum





System Parameters

A. M = mass of cart 0.5 $[kg]$

B. m = mass of the pendulum 0.2 $[kg]$

C. b = coefficient of friction for cart 0.1 $[N/m/s]$

D. l = length of pendulum center of mass 0.3 $[m]$

E. I = mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

F. F = force applied to the cart

G. x = cart position coordinate

H. θ = pendulum angle from vertical (down)

I. $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$
l\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}
$$

$$
(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u
$$

State-space:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

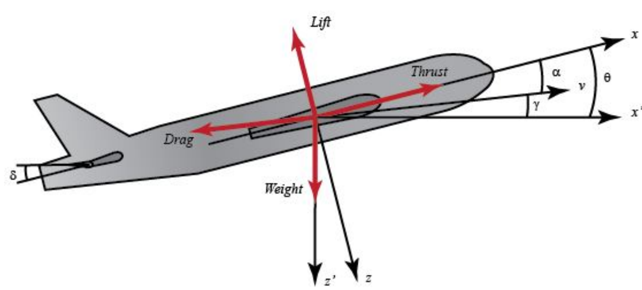
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$x[0] = \begin{bmatrix} -3 \\ 2 \\ \frac{\pi}{8} \\ -\frac{\pi}{4} \end{bmatrix}$$

vi. Aircraft Pitch



System Parameters

α = angle of attack	q = pitch rate
θ = pitch angle	δ = elevator deflection angle
$\mu = \frac{\rho S \bar{c}}{4m}$	ρ = air density
S = area of wing	$\bar{c}$ = mean chord length
m = aircraft mass	$\Omega = \frac{2U}{\bar{c}}$
U = equilibrium flight of speed	C_T = Coefficient of Thrust
C_D = Coefficient of Drag	C_L = Coefficient of Lift
C_W = Coefficient of Weight	C_M = Coefficient of Pitch Moment
γ = Flight path angle	$\sigma = \frac{1}{1+\mu C_L}$ = constant
i_{yy} = normalized moment of inertia	$\eta = \mu \sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma\left[-\left(C_L + C_D\right)\alpha + \frac{1}{\left(\mu - C_L\right)}q - \left(C_W \sin\gamma\right)\theta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M\left(1 - \mu C_L\right)\right]q + \left(\eta C_W \sin\gamma\right)\delta\right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$

$$x[0] = \begin{bmatrix} \frac{\pi}{16} \\ -\frac{\pi}{8} \\ \frac{\pi}{12} \end{bmatrix}$$