AE 513 - Multivariable Control - Autumn 2019

Homework 4

__Due Date__: Friday, Nov 1^{st}, 2019 at 11:59pm
Consider a system of the form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with $A \in R^{n \times n}$, $B \in R^{n \times m}$ and $C \in R^{o \times n}$. **(PTS: 0-2)** Write out the dynamics for an estimator state $\hat{x}$ with an observer gain of $L \in R^{n \times o}$ and then **(PTS: 0-2)** write out the combined dynamics for the true state x and estimated state $\hat{x}$ when a control input of $u = \bar{u} + K\hat{x}$ is applied with feedback gain $K \in R^{m \times n}$. (You can write the full dynamics in terms of the error in the estimate $e = \hat{x} - x$ if you prefer.) For each of the systems listed below perform the following steps. Feel free to use MATLAB or Python.

1. **Feedback Control: Gain Design**

   Design the feedback gain $K \in R^{m \times n}$ to stabilize the system matrix.

   For single input systems:

   (a) **(PTS: 0-2)** Compute the characteristic polynomial of A.

   $$\det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$$

   Select (distinct) desired eigenvalues $\lambda_1, \ldots, \lambda_n$ for the closed loop system $A+BK$ so that the closed loop system will be stable. Compute the desired characteristic polynomial for $A+BK$ using the formula

   $$\det(\lambda I - (A + BK)) = \Pi_i(\lambda - \lambda_i) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1 \lambda + \beta_0$$

   (b) **(PTS: 0-2)** If the system is controllable, compute a coordinate transformation $x = Tz$ such that the system in the z coordinates is in *controllable canonical form*

   $$\dot{z} = \bar{A}z + \bar{B}u$$

   where

   $$\bar{A} = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & \alpha_1 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Use the fact that if T exists, then the controllability matrix in the two different coordinates are related by

$$\begin{bmatrix} \bar{A}^{n-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix} = T^{-1} \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \tag{1}$$

(Double check that $\bar{A} = T^{-1}AT$ and $\bar{B} = T^{-1}B$).

(c) **(PTS: 0-2)** Compute the gain matrix $\bar{K}$ such that $\bar{A} + \bar{B}\bar{K}$ has the desired characterisitic polynomial. $\lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0$.

(d) **(PTS: 0-2)** Compute the feedback gain matrix K so that the closed loop system matrix $A + BK$ has the desired characteristic polynomial using $\bar{K}$ and T.

For multi-input systems:

(PTS: 0-2) Instead of the above steps, use the `place` command in MATLAB (or Python) to design the feedback gain matrix K.

2. **Feedback Control: Conditioning of Closed-Loop Eigenvectors**

(a) **(PTS: 0-2)** Compute the right eigenvectors of the closed loop matrix $A + BK$ and place them in the columns of a matrix X. Consider the closed-loop system transformed into the eigenvector coordinates $x = Xz$.

$$\begin{aligned} & & \dot{x} &= (A + BK)x \\ &\Rightarrow & \dot{z} &= \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} z \end{aligned}$$

(b) **(PTS: 0-2)** For an initial condition $x(0)$, note that the initial condition in the eigenvector coordinates is given by $z(0) = X^{-1}x(0)$. Also note that the magnitude of $z(0)$ is given by

$$|z(0)| = \sqrt{x(0)^*(X^{-1})^*X^{-1}x(0)}$$

. where * is the *conjugate tranpose* (performs the regular transpose as well as conjugates any complex numbers, ie. negates the imaginary parts).

Compute the minimum and maximum eigenvalues of $M = (X^{-1})^*X^{-1}$, $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ and their corresponding eigenvectors $x_{\min}$ and $x_{\max}$. Be sure to normalize $x_{\min}$ and $x_{\max}$ to have the same magnitude. Note that $x_{\min}$ and $x_{\max}$ correspond to the initial condition directions that have the minimum and maximum magnitudes in the eigenvector coordinates.

(c) **(PTS: 0-2)** Note that the *singular values* of X^{-1}, usually denoted $\sigma_1, \ldots, \sigma_n$ are the square roots of the eigenvalues of M. Compute the *condition number* of X^{-1}

$$\kappa(X^{-1}) = \frac{\sigma_{\max}(X^{-1})}{\sigma_{\min}(X^{-1})} = \frac{\sqrt{\lambda_{\max}(M)}}{\sqrt{\lambda_{\min}(M)}}$$

Assuming that $x_{\max}$ and $x_{\min}$ are both normalized to have the same length, how does the condition number relate to the ratio $\frac{|X^{-1}x_{\max}|}{|X^{-1}x_{\min}|}$?

3. **Observer: Gain Design**

   Repeat the steps from Part 1, to design the observer gain $L \in R^{n\times o}$. Use the same technique you used to design K, only considering the closed-loop matrix $A^T + C^T L^T$. (For single output systems, compute the transformation explicitly using the controllable canonical form method; for multi-output systems use the `place` command.) Note that you should choose eigenvalues for the observer that converge significantly faster than the eigenvalues you chose for the controller gain so that the state estimate converges faster than the true state.

4. **Observer: Conditioning of Closed-Loop Eigenvectors**

   Consider the closed loop error dynamics

   $$\dot{e} = (A + LC)e$$

   Repeat the steps from Part 2 to compute the eigenvectors of $A+LC$ and find the initial error conditions e_{max} and e_{min} that have the maximum and minimum norms in the eigenvector coordinates of $A + LC$.

5. **Simulations**

   Simulate the closed-loop dynamics of either $[x, \hat{x}]^T$ (or $[x, e]^T$) in the following scenarios. You can use either `ode45` or transform the system into discrete time. For each case, plot the state, estimated state, and control trajectories.

   (a) **(PTS: 0-2)** Set $\bar{u} = 0$ and initialize the state $x(0) = x_{\text{max}}$ and $\hat{x}(0) = x_{\text{max}} + e_{\text{max}}$.

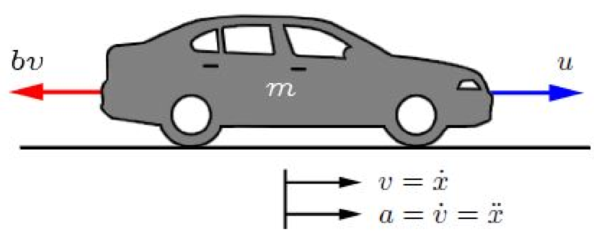
   (b) **(PTS: 0-2)** Set $\bar{u} = 0$ and initialize the state $x(0) = x_{\text{min}}$ and $\hat{x}(0) = x_{\text{min}} + e_{\text{min}}$. Make sure that x_{max} and x_{min} are normalized to have the same length and make sure that e_{max} and e_{min} have the same length. **(PTS: 0-2)** Compare the results with the previous simulation and discuss.

   (c) **(PTS: 0-2)** Set $\bar{u} = \gamma_1 \sin(\omega_1 t)$ for single input systems and $\bar{u} = [\gamma_1 \sin(\omega_1 t), \gamma_2 \sin(\omega_2 t)]$ for multi-input systems and initialize $x(0) = 0$, $\hat{x}(0) = x(0) + e_{\text{max}}$. **(PTS: 0-2)** Experiment with different amplitudes γ_1, γ_2 and different frequencies ω_1, ω_2 and discuss.

Systems

1.

   Cruise Control



2.

System Parameters

(a) $m =$ vehicle mass 1000 kg

(b) $b =$ damping 50 N.s/m

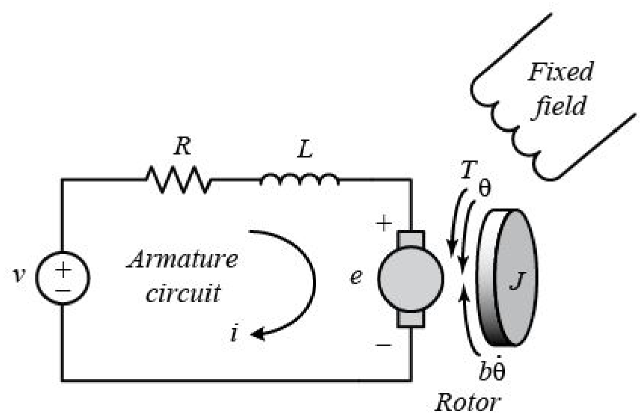
Equations of Motion:

$$m\dot{v} + bv = u$$
$$y = v$$

State-space:

$$[\dot{v}] = \left[\frac{-b}{m}\right][v] + \left[\frac{1}{m}\right][u]$$
$$y = [1][v]$$
$$[\dot{v}] = \left[-0.05\right]\left[v\right] + \left[0.001\right]\left[u\right]$$
$$y = \left[1\right]\left[v\right]$$
$$x[0] = 10$$

3. DC Motor Position



System Parameters

(a) $J =$ moment of inertia of the rotor $3.2284e^{-6}$ $[kg \cdot m^2\ ]$

(b) $b =$ motor viscous friction constant $3.5077e^{-6}$ $[N \cdot m \cdot s]$

(c) $K_e =$ electromotive force constant 0.0274 $[V/rad/s]$

(d) $K_t =$ motor torque constant 0.0274 $[N \cdot m/Amp]$

(e) $R =$ electric resistance 4.0 $[Ohm]$

(f) $L =$ electric inductance $2.75e^{-6}$ [H]

Equations of Motion:

$$T = K_t i$$
$$e = K_e \dot{\theta}$$
$$J\ddot{\theta} + b\dot{\theta} = Ki$$
$$L\frac{di}{dt} + Ri = V - K\dot{\theta}$$

State-space:

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{b}{J} & \frac{K}{J}\\ 0 & -\frac{K}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}V$$

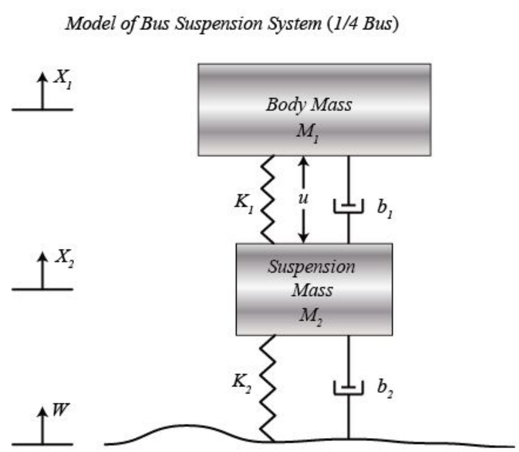
$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}$$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{i}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -1.087 & 8487\\ 0 & -9964 & -1.455e^{6}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix} + \begin{bmatrix}0\\ \\ 3.636e^{5}\end{bmatrix}V$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ i\end{bmatrix}$$

$$x[0] = \begin{bmatrix}\frac{\pi}{2}\\ -\frac{\pi}{8}\\ 2\end{bmatrix}$$

4. Suspension



System Parameters

(a) M_1 = 1/4 bus body mass 2500 $[kg]$

(b) M_2 = suspension mass 320 $[kg]$

(c) K_1 = spring constant of suspension system $80,000$ $[N/m]$

(d) K_2 = spring constant of wheel and tire $500,000$ $[N/m]$

(e) b_1 = damping constant of suspension system 350 $[N \cdot s/m]$

(f) b_2 = damping constant of wheel and tire $15,020$ $[N \cdots /m]$

(g) U = control force

Equations of Motion:

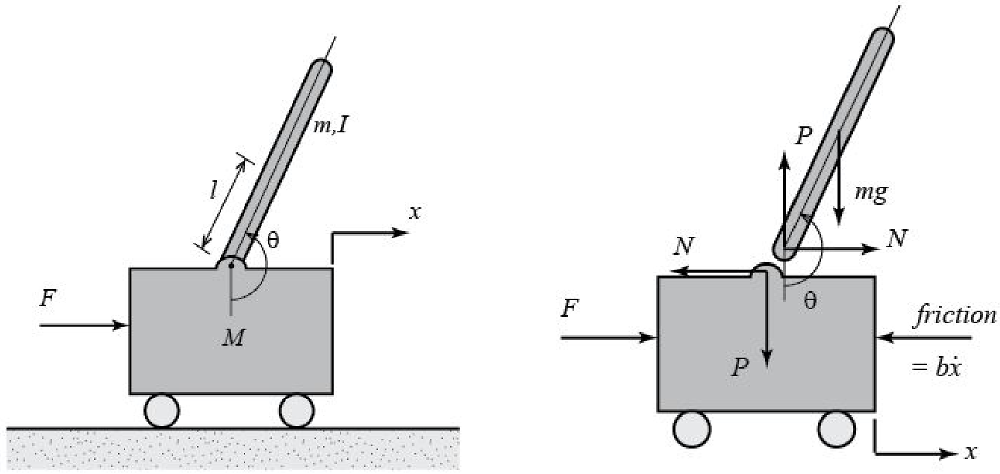
$$M_1\ddot{X}_1 = -b_1\left(\dot{X}_1 - \dot{X}_2\right) - K_1\left(X_1 - X_2\right) + U$$
$$M_2\ddot{X}_2 = b_1\left(\dot{X}_1 - \dot{X}_2\right) + K_1\left(X_1 - X_2\right) + b_2\left(\dot{W} - \dot{X}_2\right) + K_2\left(W - X_2\right) - U$$

State-space:

$$\begin{bmatrix} \dot{X}_1 \\ \ddot{X}_1 \\ \dot{X}_2 \\ \ddot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-b_1 b_2}{M_1 M_2} & 0 & \left[\frac{b_1}{M_1}\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) - \frac{K_1}{M_1}\right] & \frac{-b_1}{M_1} \\ \frac{b_2}{M_2} & 0 & -\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) & 1 \\ \frac{K_2}{M_2} & 0 & -\left(\frac{K_1}{M_1} + \frac{K_1}{M_2} + \frac{R_2}{M_2}\right) & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & \frac{b_1 b_2}{M_1 M_2} \\ 0 & \frac{-b_2}{M_2} \\ \left(\frac{1}{M_1} + \frac{1}{M_2}\right) & \frac{-K_2}{M_2} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6.571 & 0 & -25.26 & -0.14 \\ 46.94 & 0 & -48.17 & 1 \\ 1563 & 0 & -1845 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.0004 & 6.571 \\ 0 & -46.94 \\ 0.003525 & -1563 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$x[0] = \begin{bmatrix} 0.5 \\ 2 \\ -0.5 \\ -3 \end{bmatrix}$$

5. Inverted Pendulum



System Parameters

(a) M = mass of cart 0.5 $[kg]$

(b) m = mass of the pendulum 0.2 $[kg]$

(c) b = coefficient of friction for cart 0.1 $[N/m/s]$

(d) l = length of pendulum center of mass 0.3 $[m]$

(e) I = mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

(f) F = force applied to the cart

(g) x = cart position coordinate

(h) θ = pendulum angle from vertical (down)

(i) $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\big(I + ml^2\big)\,\ddot{\phi} - mgl\phi = ml\ddot{x}$$
$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

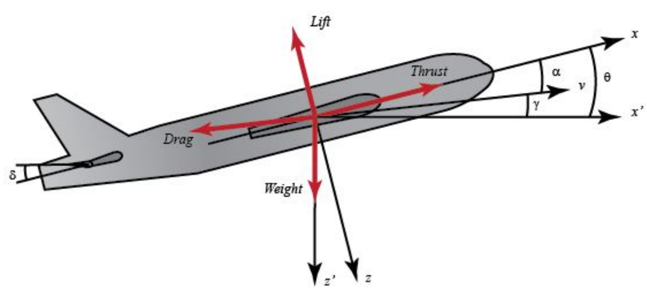
$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$x[0] = \begin{bmatrix} -3 \\ 2 \\ \frac{\pi}{8} \\ -\frac{\pi}{4} \end{bmatrix}$$

6. Aircraft Pitch



System Parameters

α = angle of attack	q = pitch rate
θ = pitch angle	δ = elevator deflection angle
$\mu = \frac{\rho S\bar{c}}{4m}$	ρ = air density
S = area of wing	$\bar{c}$ = mean chord length
m = aircraft mass	$\Omega = \frac{2U}{\bar{c}}$
U = equilibrium flight of speed	C_T = Coefficient of Thrust
C_D = Coefficient of Drag	C_L = Coefficient of Lift
C_W = Coefficient of Weight	C_M = Coefficient of Pitch Moment
γ = Flight path angle	$\sigma = \frac{1}{1+\mu C_L}$ = constant
i_{yy} = normalized moment of inertia	$\eta = \mu\sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma\left[-\left(C_L + C_D\right)\alpha + \frac{1}{(\mu - C_L)}q - \left(C_W \sin\gamma\right)\theta + C_L\right]$$
$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M\left(1 - \mu C_L\right)\right]q + \left(\eta C_W \sin\gamma\right)\delta\right]$$
$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$
$$x[0] = \begin{bmatrix} \frac{\pi}{16} \\ -\frac{\pi}{8} \\ \frac{\pi}{12} \end{bmatrix}$$