AE 513 - Multivariable Control - Autumn 2019

Homework 5

<u>**Due Date**</u>: Friday, Nov 8^{st}, 2019 at 11:59pm

1. **Lagrange Multipliers** Use the method of Lagrange multipliers to solve the following optimization problems.

   (a) **(PTS: 0-2)**

$$\begin{aligned} \min_{x_1,x_2} \quad & x_1^2 + 2x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 = 1; \end{aligned}$$

   (b) **(PTS: 0-2)**

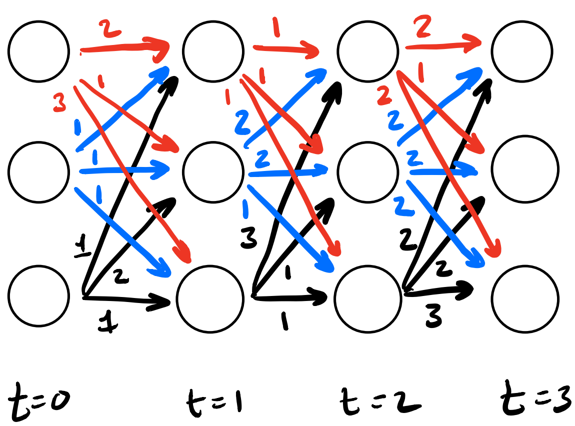
$$\begin{aligned} \min_{x \in R^3} \quad & x^\intercal Q x + c^\intercal x \\ \text{s.t.} \quad & Ax = b; \end{aligned}$$

   with

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

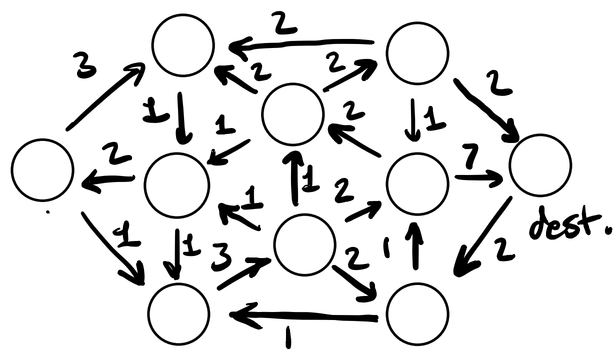
2. **Dynamic Programming**

   (a) **Finite horizon example:**
   (PTS: 0-2) Compute the minimal cost from layer $t = 0$ to layer $t = 3$ for each node by computing the optimal *cost-to-go* or *value* at each node.



(b) **Fixed point example:**

(PTS: 0-2) Consider the graph below. Initialize the *cost-to-go* or *value* at each node at infinity (or some large number). Iterate over each node using the Bellman equation to update the value to reflect the minimum cost-to-go to the destination. (During each update, you should assume that the values at the other nodes accurately reflect the cost-to-go to the destination.) Repeat this process until the value at each node converges.



3. Finite Horizon LQR Extensions

Consider the following extension of the LQR problem in the continuous time case

$$\min_{u(t)} \quad \int_0^T (x-\bar{x})^\intercal Q(t)(x-\bar{x}) + 2(x-\bar{x})^\intercal N(t)(u-\bar{u}) + (u-\bar{u})^\intercal R(t)(u-\bar{u}) \; dt + \big(x(T)-\bar{x}(T)\big)^\intercal Q(T)\big(x(T)-\bar{x}(T)\big) \tag{1}$$

$$\text{s.t.} \quad \dot{x} = A(t)x + B(t)u, \quad x(0) = x_0 \tag{2}$$

The time dependence on x and u are assumed. $\bar{u}(t)$ and $\bar{x}(t)$ are desired control and state trajectories respectively, and $A \in R^{n\times n}$ $B \in R^{n\times m}$, $Q \in R^{n\times n}$, $N \in R^{n\times m}$ and $R \in R^{m\times m}$. The equivalent discrete time problem is given by

$$\min_{u(t)} \quad \sum_{t=0}^{T-1} (x-\bar{x})^\intercal Q[t](x-\bar{x}) + 2(x-\bar{x})^\intercal N[t](u-\bar{u}) + (u-\bar{u})^\intercal R[t](u-\bar{u}) \; dt + \big(x[T]-\bar{x}[T]\big)^\intercal Q[T]\big(x[T]-\bar{x}(T)\big) \tag{3}$$

$$\text{s.t.} \quad x[t+1] = A[t]x[t] + B[t]u[t], \quad x[0] = x_0 \tag{4}$$

(a) In the discrete time case:

i. **(PTS: 0-2)** Show that the cost-to-go from time t has the form $(x-\bar{x})^\intercal P[t](x-\bar{x})$ and derive the Riccati update equation to solve for $P[t]$.

ii. **(PTS: 0-2)** Derive the form of the optimal feedback control law.

(b) In the continuous time case:

i. **(PTS: 0-2)** Show that the cost-to-go from time t has the form $(x-\bar{x})^\intercal P(t)(x-\bar{x})$ and derive the Riccati differential equation to solve for $P(t)$.

ii. **(PTS: 0-2)** Derive the form of the optimal feedback control law.

4. **Infinite Horizon LQR**

The *infinite horizon continuous time LQR problem* is given by

$$\min_{u(t)} \quad \int_0^\infty x^\intercal Q x + u^\intercal R u \; dt \tag{5}$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{6}$$

For a long time horizon, the cost-to-go matrix P converges, ie. $\dot{P}$ goes to 0 and can be found by solving the *algebraic Riccati equation*

$$0 = A^\intercal P + PA + Q - PBR^{-1}B^\intercal P$$

with optimal gains given by

$$K = -R^{-1}B^\intercal P$$

Let $x \in R^3$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Select $Q = Q^\intercal \geq 0$ and $R = R^\intercal > 0$.

(PTS: 0-2) Solve the Riccati differential equation

$$-\dot{P} = A^\intercal P + PA + Q - PBR^{-1}B^\intercal P$$

from the terminal condition $P(T) = Q$.

(PTS: 0-2) Demonstrate that as P progresses backwards in time it converges to the solution to the algebraic Riccati equation. (You can solve the algebraic Riccati equation using the `lqr` function in MATLAB.)

5. **LQR Design**

For the two systems given below perform the following steps. You can use either discrete or continuous time.

(a) **(PTS: 0-2)** Set $Q = C^\intercal C$ and pick R. Compute an infinite horizon LQR controller.

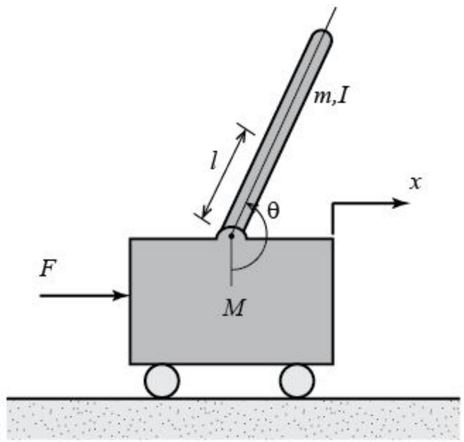
(b) **(PTS: 0-2)** Compute a finite horizon LQR controller.

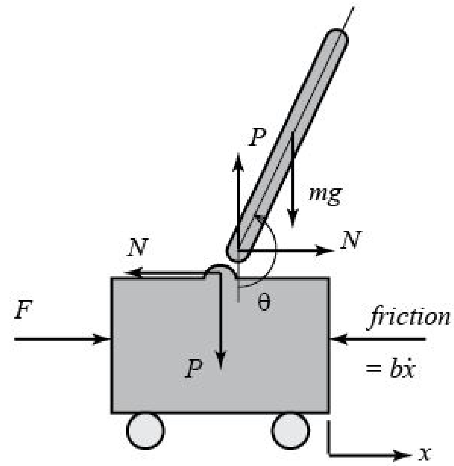
(c) **(PTS: 0-2)** Simulate the system from several initial conditions for the two different controllers and compare.

(d) **(PTS: 0-2)** For the infinite horizon controller, modify Q to penalize different states. Try several cases. Recompute the optimal controller in each case. (You can use the `lqr` command.) Simulate the trajectories and compare them.

Systems

1. Inverted Pendulum





System Parameters

(a) $M =$ mass of cart 0.5 $[kg]$

(b) $m =$ mass of the pendulum 0.2 $[kg]$

(c) $b =$ coefficient of friction for cart 0.1 $[N/m/s]$

(d) $l =$ length of pendulum center of mass 0.3 $[m]$

(e) $I =$ mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

(f) $F =$ force applied to the cart

(g) $x =$ cart position coordinate

(h) $\theta =$ pendulum angle from vertical (down)

(i) $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}$$
$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

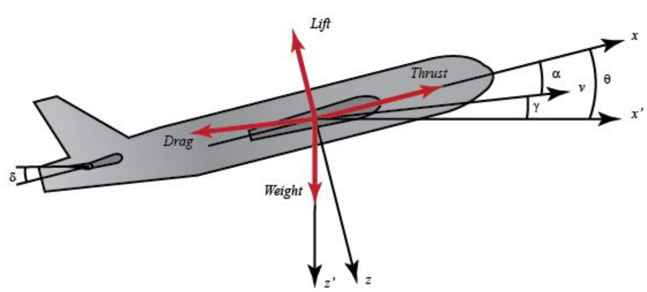
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

2. Aircraft Pitch



System Parameters

α = angle of attack
θ = pitch angle
$\mu = \frac{\rho S \bar{c}}{4m}$
S = area of wing
m = aircraft mass
U = equilibrium flight of speed
C_D = Coefficient of Drag
C_W = Coefficient of Weight
γ = Flight path angle
i_{yy} = normalized moment of inertia

q = pitch rate
δ = elevator deflection angle
ρ = air density
$\bar{c}$ = mean chord length
$\Omega = \frac{2U}{\bar{c}}$
C_T = Coefficient of Thrust
C_L = Coefficient of Lift
C_M = Coefficient of Pitch Moment
$\sigma = \frac{1}{1+\mu C_L}$ = constant
$\eta = \mu \sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma \left[-\left(C_L + C_D\right)\alpha + \frac{1}{\left(\mu - C_L\right)} q - \left(C_W \sin\gamma\right)\theta + C_L \right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}} \left[\left[C_M - \eta\left(C_L + C_D\right) \right]\alpha + \left[C_M + \sigma C_M \left(1 - \mu C_L\right) \right] q + \left(\eta C_W \sin\gamma\right)\delta \right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$