AE 513 - Multivariable Control - Autumn 2019

Homework 6

Due Date: Thurs, Nov 14^{th}, 2019 at 11:59pm

1. **Laplace Transform**

   The Laplace transform of a function $f(t)$ is given by

   $$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\, dt$$

   Choose three of the following Laplace Transforms to compute:

   (a) **(PTS: 0-2)** Delta function: $\mathcal{L}\left(\delta(t)\right) = ?$

   (b) **(PTS: 0-2)** Differentiation: $\mathcal{L}\left(\dot{f}(t)\right) = ?$

   (c) **(PTS: 0-2)** Integration: $\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = ?$

   (d) **(PTS: 0-2)** Frequency Shift: $\mathcal{L}\left(e^{at}f(t)\right) = ?$

   (e) **(PTS: 0-2)** Convolution: $\mathcal{L}\left(\int_0^t g(t-\tau)f(\tau)\, d\tau\right) = ?$

2. **Transfer Functions**

   Consider the continuous time linear system

   $$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

   (PTS: 0-2) Compute the Laplace transform of the output $y(t)$, ie. $\mathcal{L}(y(t)) = Y(s)$. Your solution should be in terms of the $A, B, C, D, x_0, U(s)$, where $U(s)$ is the Laplace transform of $u(t)$.

3. **Matrix Identities**

   The Woodbury matrix identity is given by

   $$(A + UCV)^{-1} = A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}$$

   $A \in R^{n\times n}$ invertible, $C \in R^{m\times m}$ invertible, $U \in R^{n\times m}$, $V \in R^{m\times n}$.

   Side Note: If $m < n$, ie. C is smaller than A, this formula provides a way to update the inverse of a matrix A that has a low rank matrix UCV added to it without having to computing the full $n \times n$ inverse, $(A + UCV)^{-1}$. Assuming we already know A^{-1}, we can compute $(A + UCV)^{-1}$ simply by computing C^{-1} and then $(C^{-1} + VA^{-1}U)^{-1}$. If $m \ll n$, for example if $C \in R^{1\times 1}$, this operation is much cheaper computationally.

   (a) For an invertible matrix M, show that

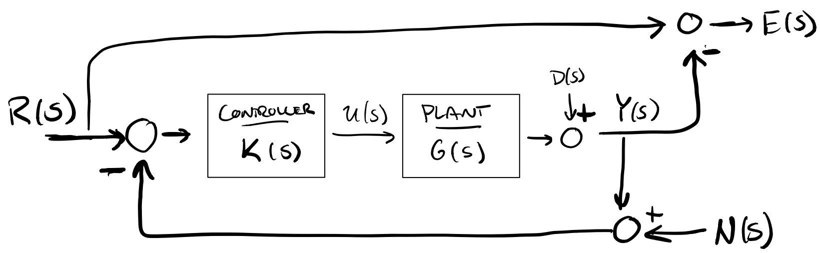
   $$(I + M)^{-1} = I - (I + M)^{-1}M$$

(b) For an invertible matrix M, show that

$$M(I+M)^{-1} = (I+M)^{-1}M$$

4. **Sensitivity Functions**

Consider the block diagram



where

(a) $G(s)$ is the plant transfer function.

(b) $K(s)$ is the plant transfer function.

(c) $U(s)$ is the control signal.

(d) $R(S)$ is the reference signal.

(e) $Y(S)$ is the output signal.

(f) $D(S)$ is the disturbance.

(g) $E(S)$ is the error between the output and reference

Show that

(PTS: 0-2) $$Y(s) = S(s)D(s) + T(s)\Big(R(s) - N(s)\Big) \quad (1)$$

(PTS: 0-2) $$E(s) = S(s)\Big(R(s) - D(s)\Big) + T(s)N(s) \quad (2)$$

where

$$S(s) = \Big(I + G(s)K(s)\Big)^{-1} \qquad \text{Sensitivity}$$

$$T(s) = G(s)K(s)\Big(I + G(s)K(s)\Big)^{-1} \qquad \text{Complementary Sensitivity}$$

Note: use the matrix identities from the previous problem.

(a) **(PTS: 0-2)** What is the effect of large $S(s)$ on the output $Y(s)$ and the error $E(s)$?

(b) **(PTS: 0-2)** What is the effect of large $T(s)$ on the output $Y(s)$ and the error $E(s)$?

5. **Kalman Filter and LQG Controller**

Consider the dynamics of the form

$$\dot{x} = Ax + Bu + w, \qquad w \sim \mathcal{N}(0, W)$$
$$y = Cx + v, \qquad v \sim \mathcal{N}(0, V)$$

where w and v are white noise processes with mean 0 and covariances W and V respectively. For each system given below, perform the following steps. You can use either continuous or discrete time.

(a) **(PTS: 0-2)** Compute the infinite-horizon LQR controller gain K with $Q(t) = I$ and $R(t) = I$. (You can use the `lqr` function in Matlab.)

(b) **(PTS: 0-2)** Implement the Kalman filter update to compute a state estimate $\hat{x}(t)$. Assume $E[x(0)] = 0$. Assume the covariance matrices W, V stay constant over time and have the form

$$W = MDM^T, \qquad V = NEN^T$$

where M and N are orthonormal matrices and D and E are diagonal matrices of the form

$$D = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n^2 \end{bmatrix}, \qquad E = \begin{bmatrix} \tau_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tau_o^2 \end{bmatrix}$$

M and N will be given. Initially, assume $\sigma_i = 1$ and $\tau_j = 1$ for all i, j. You will vary σ_i and τ_j in Part (f). (Refer to Specification)

(c) **(PTS: 0-2) OPTIONAL** What is an interpretation of the columns of M and N?

(d) **(PTS: 0-2)** Track the system trajectory for the following inputs.

i. Step Input: $u = 1 + K\hat{x}$

ii. Sine wave: $u = sin(t) + K\hat{x}$

Plot the state and estimated state (or error) trajectories.

(e) **(PTS: 0-2)** Compute the steady state Kalman gain and compare it's performance with the time varying gain update from Part (b) by plotting the trajectories.

(f) **(PTS: 0-2)** Vary the covariance matrices by adjusting the values of σ_i and τ_i. Plot the trajectories.

(g) **(PTS: 0-2)** Comment on how varying the covariance matrices changes the performance of the filter.

Pendulum Specification

$$dt = 0.02, \qquad x_0 = \begin{bmatrix} 1 & -0.5 & \frac{\pi}{16} & -\frac{\pi}{32} \end{bmatrix}^\mathsf{T}$$

$$W = MDM^T, \qquad V = NEN^T$$

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = 10^{-4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad E = 10^{-4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Aircraft Specification

$$dt = 0.02, \qquad x_0 = \begin{bmatrix} \frac{\pi}{16} & \frac{\pi}{64} & -\frac{\pi}{32} \end{bmatrix}^{\mathsf{T}}$$

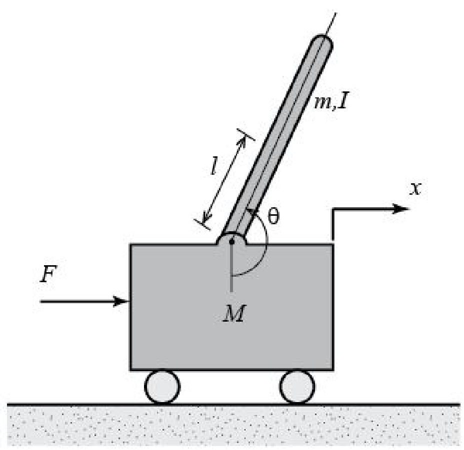
$$W = MDM^T, \qquad V = NEN^T$$

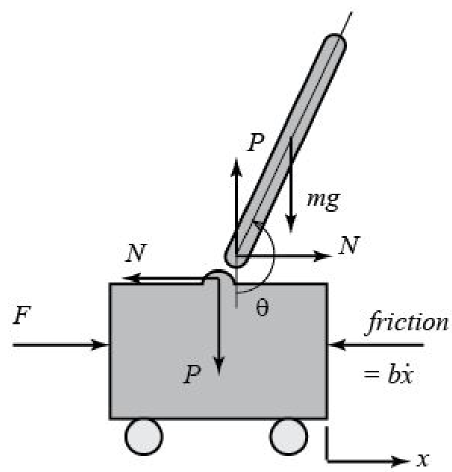
$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 \end{bmatrix}$$

$$D = 10^{-6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E = 10^{-6} \begin{bmatrix} 1 \end{bmatrix}$$

Systems

(a) Inverted Pendulum





System Parameters

i. M = mass of cart 0.5 $[kg]$

ii. m = mass of the pendulum 0.2 $[kg]$

iii. b = coefficient of friction for cart 0.1 $[N/m/s]$

iv. l = length of pendulum center of mass 0.3 $[m]$

v. I = mass moment of inertia of the pendulum 0.006 $[kg \cdot m^2]$

vi. F = force applied to the cart

vii. x = cart position coordinate

viii. θ = pendulum angle from vertical (down)

ix. $\phi = \theta - \pi$

Equations of Motion (for small θ):

$$l\left(I + ml^2\right)\ddot{\phi} - mgl\phi = ml\ddot{x}$$

$$(M + m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = u$$

State-space:

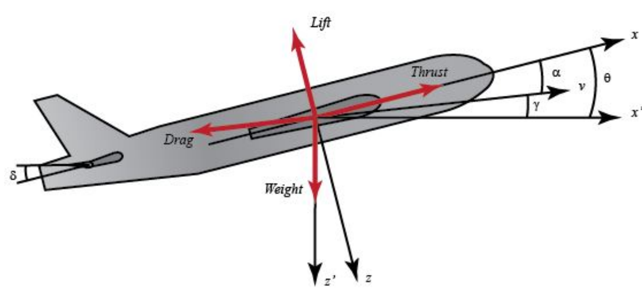
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I+ml^2}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.6727 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.4545 & 31.1818 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8182 \\ 0 \\ 4.5455 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

(b) Aircraft Pitch



System Parameters

α = angle of attack	q = pitch rate
θ = pitch angle	δ = elevator deflection angle
$\mu = \frac{\rho S \bar{c}}{4m}$	ρ = air density
S = area of wing	$\bar{c}$ = mean chord length
m = aircraft mass	$\Omega = \frac{2U}{\bar{c}}$
U = equilibrium flight of speed	C_T = Coefficient of Thrust
C_D = Coefficient of Drag	C_L = Coefficient of Lift
C_W = Coefficient of Weight	C_M = Coefficient of Pitch Moment
γ = Flight path angle	$\sigma = \frac{1}{1+\mu C_L}$ = constant
i_{yy} = normalized moment of inertia	$\eta = \mu\sigma C_M$ = constant

Equations of Motion:

$$\dot{\alpha} = \mu\Omega\sigma\left[-\left(C_L + C_D\right)\alpha + \frac{1}{\left(\mu - C_L\right)}q - \left(C_W \sin\gamma\right)\theta + C_L\right]$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}\left[\left[C_M - \eta\left(C_L + C_D\right)\right]\alpha + \left[C_M + \sigma C_M\left(1 - \mu C_L\right)\right]q + \left(\eta C_W \sin\gamma\right)\delta\right]$$

$$\dot{\theta} = \Omega q$$

State-space:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} [\delta]$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \end{bmatrix}$$