AE 513 - Multivariable Control - Autumn 2019

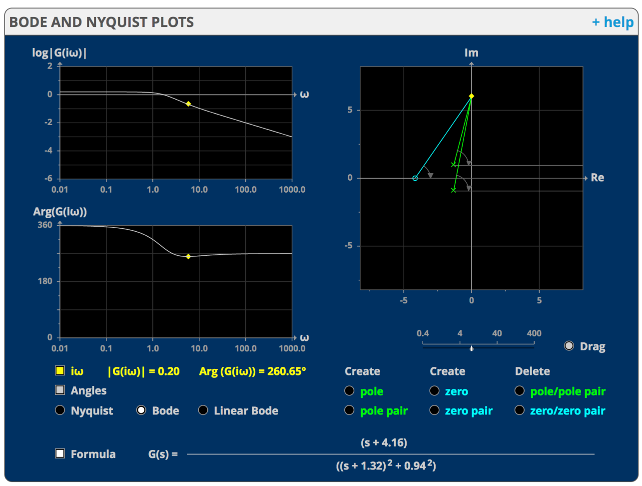
Homework 7

Due Date: Tues, Nov 26^{th}, 2019 at 11:59pm

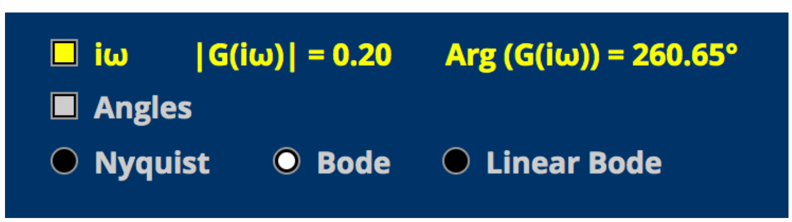
1. **Bode and Nyquist Plots**

   Use the tool in the link below to visualize the Bode plots for the transfer functions listed below.

   https://mathlets.org/mathlets/bode-and-nyquist-plots/



   Make sure to try out the $i\omega$ and **Angles** options.



   For each case, **take at least one screenshot and comment** on the behavior of the transfer function and how it is represented in the Bode plots.

   Note: The goal is to get intuition on how poles and zeros affect transfer function behavior. Make sure you spend as much or more time playing around with it as you do taking screenshots.

   (a) **(PTS: 0-2)** One real pole. Vary λ along the real axis. Specifically note what happens when $\lambda = 0$.

$$G(s) = \frac{1}{s - \lambda}$$

(b) **(PTS: 0-2)** A pair of complex poles. Vary the poles and specifically note what happens when they cross the $j\omega$-axis.

$$G(s) = \frac{1}{(s-\lambda_1)(s-\lambda_2)}$$

(c) **(PTS: 0-2)** One real pole and one real zero. Vary the zero z along the real axis. In particular note, the high frequency behavior of the transfer function.

$$G(s) = \frac{s-z}{s-\lambda}$$

(d) **(PTS: 0-2)** A pair of complex poles and a real zero. Vary the zero z along the real axis. In particular note what happens to the phase when the zero crosses the $j\omega$-axis.

$$G(s) = \frac{s-z}{(s-\lambda_1)(s-\lambda_2)}$$

(e) **(PTS: 0-2)** A pair of complex poles and a pair of complex zeros. Vary the zeros. Note what happens when the zeros cross the $j\omega$-axis.

$$G(s) = \frac{(s-z_1)(s-z_2)}{(s-\lambda_1)(s-\lambda_2)}$$

2. **Proper Transfer Functions**

Transfer functions that correspond to *causal systems* - where only past information affects the present (as opposed to future information) always have a denominator with polynomial degree greater than or equal to the numerator. These transfer functions are called *proper*.

(a) **Proper:** Denominator degree greater than or equal to numerator degree.

(b) **Strictly Proper:** . Denominator degree strictly greater than numerator degree.

Non-proper transfer functions can be used in signal processing for post-processing and smoothing signals after the fact.

(a) **(PTS: 0-2** What is the high frequency behavior of a transfer function where the degree of the numerator and denominator are equal? What is the high frequency behavior of a strictly proper transfer function?

(b) **(PTS: 0-2)**

Consider the SISO LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$, $D \in R^{1\times 1}$ where A is diagonalizable with distinct eigenvalues. Note that D transfers the input u to the output y directly with out passing it through the dynamics. Show that the transfer function representation $G(s)$ of this system is strictly proper if $D = 0$.

(c) **(PTS: 0-2)** Now consider the MIMO generalization where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{o\times n}$, $D \in R^{o\times m}$. Let $G(s)$ be the $o \times m$ dimensional matrix of transfer functions. Show that $G_{kl}(s)$ is strictly proper if $D_{kl} = 0$. (Hint: Diagonalize A.)

(d) **(PTS: 0-2)** In general, is the transfer function representation of full state feedback strictly proper or just proper?

(e) **(PTS: 0-2)** From the lecture notes, the state space representation of the continuous time steady state LQG controller for reference tracking can be written as

$$\begin{aligned}\dot{\hat{x}} &= \begin{bmatrix} A + BK + LC \end{bmatrix} \hat{x} + \begin{bmatrix} -BK & -LC \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix} \\ u &= K\hat{x} + \begin{bmatrix} -K & 0 \end{bmatrix} \begin{bmatrix} r \\ y \end{bmatrix}\end{aligned}$$

In general, are the transfer functions from the reference $r(t)$ to the control signal $u(t)$ strictly proper or just proper? What about the transfer functions from the output $y(t)$?

3. **MIMO Bode Plots**

Consider the dynamics

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and consider the transfer function

$$L(s) = C(sI - A)^{-1}B$$

Let $\bar{\sigma}(\cdot)$ be the maximum singular value and $\underline{\sigma}(L(\cdot)$ be the minimum singular value. Let

$$S(s) = (I + L(s))^{-1}, \qquad T(s) = L(s)(I + L(s))^{-1}$$

Plot the following on a log-log scale and comment. You can use the `svd` commmand in MATLAB.

(a) **(PTS: 0-2)** $\bar{\sigma}(L(j\omega))$, $\underline{\sigma}(L(j\omega))$

(b) **(PTS: 0-2)** $\bar{\sigma}(S(j\omega))$, $\underline{\sigma}(S(j\omega))$

(c) **(PTS: 0-2)** $\bar{\sigma}(T(j\omega))$, $\underline{\sigma}(T(j\omega))$

4. **Extended Kalman Filter**

Consider the following first-order nonlinear system:

$$x[t + 1] = x[t]$$

$$y[t] = \sin(x[t] \cdot t) + v[t]$$

(a) **(PTS: 0-2)** Create 201 synthetic measurements of the aforementioned system with a time step of 0.1 seconds with $x[0] = 1$ and covariance $V = 0.1$.

(b) **(PTS: 0-2)** Develop an extended Kalman filter to estimate the frequency $x[t]$.

(c) **(PTS: 0-2)** Run the filter initializing it with the initial estimates given below.

i. $\hat{x}[0] = 10, \qquad S[0] = 1$

ii. $\hat{x}[0] = 7, \qquad S[0] = 1$

iii. $\hat{x}[0] = 4, \qquad S[0] = 1$