AE 513 - Multivariable Control - Autumn 2019

Homework 8

Due Date: Mon, Dec 2^{nd}, 2019 at 11:59pm

1. **Lyapunov Equation and Stability**

   (a) **(PTS: 0-2)** Show that if A is stable, then $P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$ solves

   $$A^T P + PA + Q = 0$$

   (b) **(PTS: 0-2)** Show that if there exists $P = P^T > 0$ that solves

   $$A^T P + PA + Q = 0$$

   for $Q = Q^T > 0$, then A is stable and find a bound on the largest real part of the eigenvalues of A in terms of the eigenvalues of Q and P.

   Suppose $V(x)$ is positive definite, ie. $V(x) > 0\ \forall x \neq 0$, $V(0) = 0$, $V(x)$ has bounded level sets.

   (c) **(PTS: 0-2)** Show that if $\frac{\partial V}{\partial x} f(x) < 0$, then $x = 0$ is asymptotically stable.

   (d) **(PTS: 0-2)** Show that if $\frac{\partial V}{\partial x} f(x) < \mu V(x)$, then $x = 0$ is exponentially stable. What is an upper bound on the decay rate?

2. **Hamiltonian Systems for Riccati Equations**

   Let

   $$A = \begin{bmatrix} -2 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

   $$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 \end{bmatrix}, \qquad Q_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (a) **(PTS: 0-2)** Use the Hamiltonian system to compute the solution $P(t)$ to the Riccati differential equation

   $$-\dot{P} = A^T P + PA + Q - PBR^{-1}B^T P, \qquad P(T) = Q_T$$

   Show the calculation method and then write down $P(t)$ at several different time steps.

   (b) **(PTS: 0-2)** Use the Hamiltonian matrix to compute the solution to the algebraic Riccati equation

   $$0 = A^T P + PA + Q - PBR^{-1}B^T P$$

3. **Adjoint Method for Optimal Control**

Consider the problem of finding the shape of a slope that causes a frictionless stone to slide the farthest in a given period of time.

This problem can be formulated as

$$\max_{\substack{\theta(t) \text{ for} \\ t\in[0,T]}} \quad J = x(T)$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{v} \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ -g\sin(\theta) \end{bmatrix}}_{f(z,\theta)}, \qquad \begin{bmatrix} x(0) \\ y(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where x and y are the horizontal and vertical positions respectively, v is the velocity, $g = 9.81\ m/s^2$ is the acceleration due to gravity, the cost $J = x(T)$ is the final horizontal position, and $\theta(t)$ is the angle of the slope (from horizontal) at time t. Let $z \in R^3$ be the full state vector $z = [x\ y\ v]^T$.

(a) **(PTS: 0-2)** Write expressions for $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \theta}$.

(b) **(PTS: 0-2)** Write out the backwards differential equation for computing the co-state $p(t)$ from the terminal condition $p(T) = \frac{\partial J}{\partial z(T)} = [1\ 0\ 0]$ and the expression for computing $\frac{\partial J}{\partial \theta(t)}$ from the co-state $p(t)$.

(c) **(PTS: 0-2)** Write code to compute an ascent direction $\frac{\partial J}{\partial \theta(t)}$ for the cost J with respect to the slope at each time $\theta(t)$.

(d) **(PTS: 0-2)** Initializing $\theta(t) = 0$ for $t \in [0, T]$ (a flat slope), write code to update the slope using gradient ascent with step size α according to the equation

$$\theta^+(t) = \theta(t) + \alpha\frac{\partial J}{\partial \theta(t)}$$

until the cost converges. You can work in discrete time or continuous time. If you use discrete time, you can assume the simple discretization scheme

$$z[t+1] = z[t] + \Delta t f(z, \theta)$$

Use $\alpha = 0.01$, $\Delta t = 0.1$, and $T = 20$ seconds (200 time steps).

(e) **(PTS: 0-2)** Plot the resulting optimal slope as well as several of the suboptimal slopes from the earlier iterations.

Note Problems 4 and 5 will be discussed in class on **Tuesday (Nov 26)**. They deal with the H_2 and H_∞ design methods.

4. **Optimal Controller Synthesis (CHOOSE 4 OR 5)**

   Consider the simple harmonic oscillator (spring-mass-damper) as a generalized process G,

$$m\ddot{x} = -kx - b\dot{x} + u + \mu$$
$$y = x + \eta$$
$$\omega = \Big(\mu, \eta\Big), \qquad z = \Big(x, u\Big)$$
$$m = k = b = 1$$

   (a) Compute the H_2 and H_∞ norm of the open-loop transfer function $G_{z\omega}$, set $u = 0$, and plot step response.

   (b) Plot the open-loop sensitivity transfer function $1/(1+G_{yu})$and compute the gain, phase, and stability margins.

   (c) Synthesize an H_2- optimal state-feedback controller K_2 treating ω and z as an error to be minimized with respect to the standard $\mathcal{L}_2$ norm, compute, the H_2 norm of the closed-loop transfer function and compute the gain, phase and stability margin.

   (d) Same as Part (c) but use the H_∞ norm.

   (e) Discuss Part (a) to Part (d), do the result make sense? What are the difference between how H_2 and H_∞ hape the sensitivity transfer function and margins? Which controller would you choose? When and why?

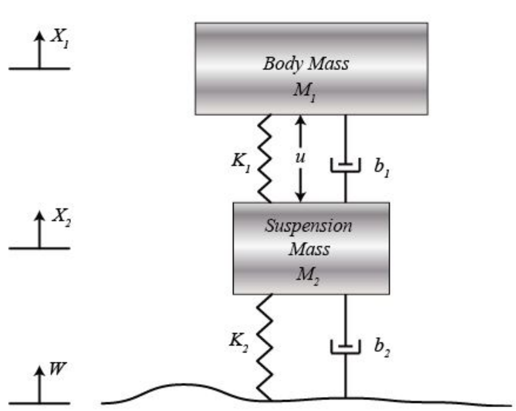
   (f) Same as Part (c) but synthesize an output-feedback controller.

   (g) Same as Part (d) but synthesize an output-feedback controller.

5. **Robust Control and Disturbance Rejection**

   Consider the suspension system.

Model of Bus Suspension System (1/4 Bus)



System Parameters

   (a) $M_1 =$ 1/4 bus body mass 2500 $[kg]$

(b) M_2 = suspension mass 320 $[kg]$

(c) K_1 = spring constant of suspension system $80,000$ $[N/m]$

(d) K_2 = spring constant of wheel and tire $500,000$ $[N/m]$

(e) b_1 = damping constant of suspension system 350 $[N \cdot s/m]$

(f) b_2 = damping constant of wheel and tire $15,020$ $[N \cdots /m]$

(g) U = control force

(h) W = road disturbance

Equations of Motion:

$$M_1\ddot{X}_1 = -b_1\left(\dot{X}_1 - \dot{X}_2\right) - K_1\left(X_1 - X_2\right) + U$$
$$M_2\ddot{X}_2 = b_1\left(\dot{X}_1 - \dot{X}_2\right) + K_1\left(X_1 - X_2\right) + b_2\left(\dot{W} - \dot{X}_2\right) + K_2\left(W - X_2\right) - U$$

State-space:

$$\begin{bmatrix} \dot{X}_1 \\ \ddot{X}_1 \\ \dot{X}_2 \\ \ddot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-b_1 b_2}{M_1 M_2} & 0 & \left[\frac{b_1}{M_1}\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) - \frac{K_1}{M_1}\right] & \frac{-b_1}{M_1} \\ \frac{b_2}{M_2} & 0 & -\left(\frac{b_1}{M_1} + \frac{b_1}{M_2} + \frac{b_2}{M_2}\right) & 1 \\ \frac{K_2}{M_2} & 0 & -\left(\frac{K_1}{M_1} + \frac{K_1}{M_2} + \frac{R_2}{M_2}\right) & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & \frac{b_1 b_2}{M_1 M_2} \\ 0 & \frac{-b_2}{M_2} \\ \left(\frac{1}{M_1} + \frac{1}{M_2}\right) & \frac{-K_2}{M_2} \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \dot{X}_1 \\ X_2 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6.571 & 0 & -25.26 & -0.14 \\ 46.94 & 0 & -48.17 & 1 \\ 1563 & 0 & -1845 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.0004 & 6.571 \\ 0 & -46.94 \\ 0.003525 & -1563 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$
$$Y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ W \end{bmatrix}$$

(a) **(PTS: 0-2)** Use state feedback setup to compute the continuous time infinite-horizon LQR controller gain K with Q and R specified below. What is the gain matrix and the norm of the close-loop transfer function?

$$Q = \begin{bmatrix} 10^3 & 0 & 0 & 0 \\ 0 & 10^5 & 0 & 0 \\ 0 & 0 & 3 \cdot 10^4 & 0 \\ 0 & 0 & 0 & 10^6 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 \end{bmatrix}$$

(b) **(PTS: 0-2)** Use state feedback setup to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller replicating the result of LQR controller in Part (a). What are the gain matrix and the norm of the close-loop transfer function?

(c) **(PTS: 0-2)** Use state feedback setup to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers minimize x_3 and u with weighting $W = 10^5$ on x_3. What are the gain matrix and the norm of the close-loop transfer function?

(d) **(PTS: 0-2)** Compare the transfer function response (Bode) of LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ from disturbance to x_3. 5 Bode plots.

(e) **(PTS: 0-2)** Simulate the closed loop system of LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ for 10s with a 0.1-m step disturbance input starting at 1s with zero initial condition and plot x_3. 5 simulations.

(f) **(PTS: 0-2)** Use output feedback setup to compute the continuous time infinite-horizon LQR controller gain K with Q and R specified in Part (a). What is the gain matrix and the norm of the close-loop transfer function?

(g) **(PTS: 0-2)** Use output feedback setup to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller replicating the result of LQR controller in Part (a). What are the gain matrix and the norm of the close-loop transfer function?

(h) **(PTS: 0-2)** Use output feedback setup to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers minimize x_3 and u with weighting $W = 10^5$ on x_3. What are the gain matrix and the norm of the close-loop transfer function?

(i) **(PTS: 0-2)** Compare the transfer function response (Bode) of LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ from disturbance to x_3. 5 Bode plots.

(j) **(PTS: 0-2)** Simulate the closed loop system of LQR, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ for 10s with a 0.1-m step disturbance input starting at 1s with zero initial condition and plot x_3. 5 simulations.