LINEAR ALGEBRA REVIEW:

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INNER PRODUCTS (DOT PRODUCTS)

$x, y \in \mathbb{R}^n$ $\qquad x \cdot y = \langle y, x \rangle = y^T x = \sum_i x_i y_i$

Norm: $|x|_2 = \sqrt{x^T x}$ $\qquad \|\cdot\| = |\cdot|$

lengths

2-norm (default) $\qquad$ other types of norms 1-norm $|\cdot|_1$

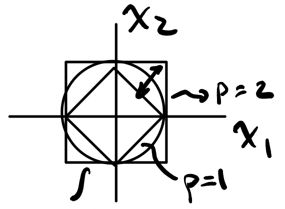
(3-norm) $|\cdot|_3$

⋮

∞-norm $|\cdot|_\infty$

$$|x|_p = \sqrt[p]{\sum_i x_i^p} \qquad |x|_1 = \sum_i |x_i|$$

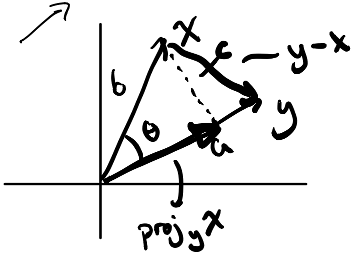
picture:



$|x|_p = 1$

$|x|_\infty = \max_i |x_i|$

angles: $\boxed{y^T x = |y||x| \cos\theta}$ ← show by applying the law of cosines to y-x



$$|c|^2 = |b|^2 + |a|^2 - 2ab\cos\theta$$

$$(y-x)^T(y-x) = x^Tx + y^Ty \boxed{-2|x||y|\cos\theta}$$

$$\hookrightarrow y^Ty \underbrace{-x^Ty - y^Tx}_{\boxed{-2x^Ty}} + x^Tx$$

if $\theta = 0$: $\cos\theta = 1$

$\underbrace{\theta = \pi/2 : \cos\theta = 0}_{\text{two vectors being orthogonal or perpendicular}}$

$$\text{proj}_y x = \underbrace{|x|\cos\theta}_{\text{projection length}} \underbrace{\frac{y}{|y|}}_{\text{unit vector in y-direction}} = \frac{1}{|y|} y^T x \frac{y}{|y|} = y\left(\frac{1}{|y|^2}\right) y^T x = \underbrace{y(y^Ty)^{-1}y^T}_{\text{general form of 1-D proj. (outer product)}} x$$

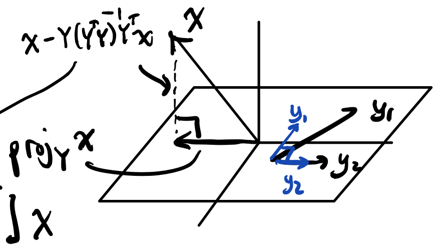
if instead 1-D projection, want to proj. onto subspace.

$$\text{proj}_Y x = \underline{Y(Y^TY)^{-1}Y^T} x$$

$$Y = \begin{bmatrix} | & & | \\ y_1 & \cdots & y_m \\ | & & | \end{bmatrix}$$

$x - Y(Y^TY)^{-1}Y^Tx$

$$\left[I - Y(Y^TY)^{-1}Y^T \right] x$$



$Y = [y_1 \, y_2]$

$Y = \underbrace{[y_1 \, y_2]}_{\text{orthonormal}}$ $Y^TY = I$

Outer Product:

inner prod: $\underbrace{y^Tx}_{\text{scalar}}$ outerprod $\underbrace{xy^T}_{\text{matrix}}$

$$xy^T = \underbrace{\begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & & \vdots \\ x_ny_1 & \cdots & x_ny_n \end{bmatrix}}_{\text{rank } 1}$$

MATRIX MULTIPLICATION:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1k} \\ \vdots & & \vdots \\ B_{n1} & & B_{nk} \end{bmatrix}$$

inner dim have to match

$$AB = \begin{bmatrix} \underline{A_{11}B_{11} + \cdots + A_{1n}B_{1n}} & \cdots & \underline{A_{11}B_{1k} + \cdots + A_{1n}B_{nk}} \\ \vdots & & \end{bmatrix}$$

slicing of inner dim of A & B have to match

interesting cases...

$$A = \begin{bmatrix} - & A_1^T & - \\ - & A_n^T & - \end{bmatrix} \quad B = \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix} \quad AB = \begin{bmatrix} A_1^TB_1 & & A_1^TB_n \\ & A_i^TB_i & \\ A_n^TB_1 & & A_n^TB_n \end{bmatrix} \leftarrow$$

pairwise inner products

now compute

$$BA = \begin{bmatrix} \downarrow & & \downarrow \\ B_1 & \cdots & B_n \end{bmatrix} \begin{bmatrix} & A_1^T & \\ & \vdots & \\ & A_n^T & \end{bmatrix} = \sum_i \lambda_i B_i A_i^T = \sum_i \left[\begin{bmatrix} B_i \end{bmatrix} \begin{bmatrix} - & A_i^T & - \end{bmatrix} \right]$$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Preview

$$A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix} = \sum_i \lambda_i v_i w_i^T$$

(more later)

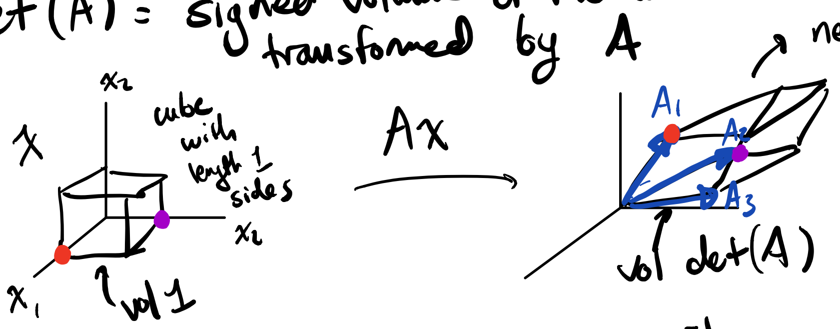
$A \quad B = [B_1 \cdots B_n] \Rightarrow AB = [AB_1 \cdots AB_n]$

vectors $x_i \rightarrow X = [x_1 --- x_{100}] \quad AX = [Ax_1 --- Ax_{100}]$

Trace & Determinant → usually for square matrices

$Tr(A) = \sum_i A_{ii} \qquad Tr(A) = Tr(A^T), \; Tr(A+B) = Tr(A) + Tr(B) \qquad Tr(AB) = Tr(BA)$ (assuming dim allow)

$det(A)$ = signed volume of the unit cube transformed by A



$A = [A_1 A_2 A_3] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

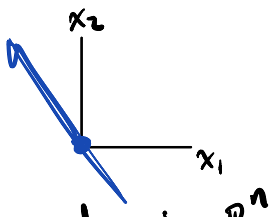
$det(A) = det(A^T) \qquad det(A^{-1}) = det(A)^{-1} \qquad det(AB) = det(BA)$
$= det(A) det(B)$

Relationship w/ eigenvalues of matrix $\quad \lambda_i$ eigenvalue

$$Tr(A) = \sum_i \lambda_i \qquad det(A) = \prod_i \lambda_i$$

RANGE, NULLSPACE, RANK:

$A \in \mathbb{R}^{m \times n}$: linear map or function $\quad n = 2, \; m = 3$



domain $\mathbb{R}^n$

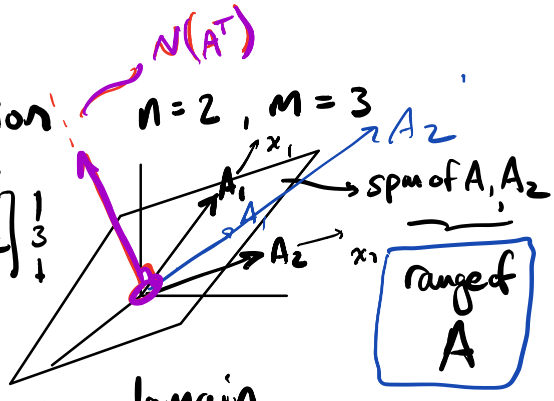
$x \in \mathbb{R}^n$

"coefficients"

$A = \underbrace{[A_1 A_2]}_{}$ ←2→ ↕3

A is "tall"

Range not the full co-domain



co-domain

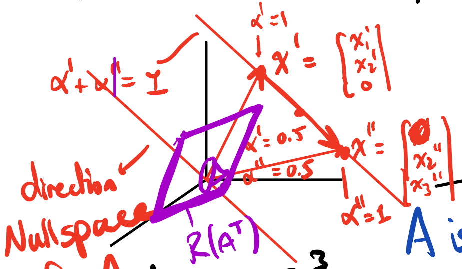
$y \in \mathbb{R}^m$

$y = A x = A_1x_1 + A_2x_2$

vectors

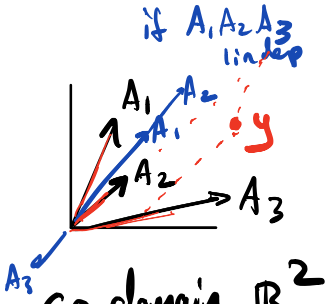
linear comb of cols A

$A = \mathbb{R}^{m \times n}$ $\quad m = 2, n = 3$



$$\overset{\leftarrow 3 \rightarrow}{A = 2\begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}}$$

A is fat



domain $\mathbb{R}^3$

$x \in$

co domain $\mathbb{R}^2$

$y \in$

$$y = Ax$$
$$= A_1x_1 + A_2x_2 + A_3x_3$$

$Ax = 0$

to clarify Nullspace defined by $x' - x''$

first soln: $y = A_1 x'_1 + A_2 x'_2$

2nd soln $y = A_2 x''_2 + A_3 x''_3$

$$x = \alpha' \begin{bmatrix} x'_1 \\ x'_2 \\ 0 \end{bmatrix} + \alpha'' \begin{bmatrix} 0 \\ x''_2 \\ x''_3 \end{bmatrix}$$

$\alpha' y + \alpha'' y = Ax$ $\qquad$ $\alpha' + \alpha'' = \underline{1}$

$$Ax' = y = Ax''$$
$$\downarrow$$
$$A(x' - x'') = 0$$

A is fat: general soln to $y = Ax$ is $x = \underset{\text{specific soln.}}{x^0} + \overset{\text{anything in } N(A)}{x_{NS}}$

$$y = A(x^0) + \underbrace{Ax_{NS}}_{0}$$

Summary:

A square invertible: unique soln

A tall: probably no solution

A fat: continuum/subspace of solns

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other comments: $x \in N(A) \rightarrow$ orthogonal to rows

$$\begin{bmatrix} -A_1^T- \\ -A_n^T- \end{bmatrix} x = \begin{bmatrix} A_1^T x \\ \vdots \\ A_n^T x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

domain:

$N(A) \perp R(A^T)$

$\text{span}\{N(A), R(A^T)\} = \text{dom.}$

$N(A) \overset{\perp}{\oplus} R(A^T) = \mathbb{R}^n$

co-domain

$N(A^T) \perp R(A)$

$\text{span}\{N(A^T), R(A)\} = \text{codom}$

$N(A^T) \overset{\perp}{\oplus} R(A) = \mathbb{R}^m$

RANK: row rank: # of lin ind rows

col rank: # of lin ind cols

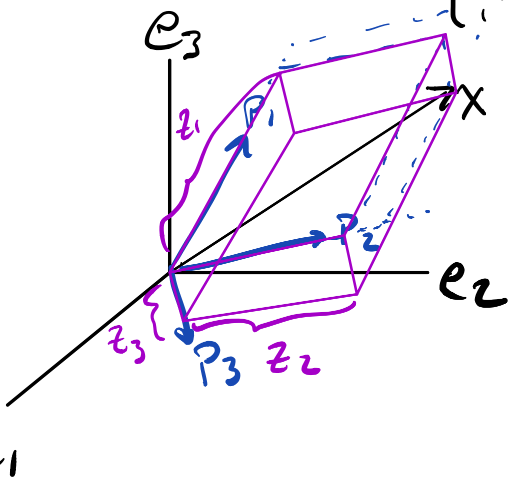
row rank = col rank = rank

Coordinates and Change of Basis:

$$\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

also

$$x = I x$$



$$x = \underbrace{\begin{bmatrix} | & | & | \\ p_1 & p_2 & p_3 \\ | & | & | \end{bmatrix}}_{\text{basis} \to P} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

coordinates of x w.r.t. P

$$x = Pz$$

↳ coord. transform

$$z = P^{-1}x$$

↳ inverse coord. transform

Now...

A acting on x: $x' = Ax$

suppose want to represent x in the P coords, ie. $z = P^{-1}x$

"how do I change A so that $x' = Pz'$ $\quad x = Pz$

ie. find B s.t. $z' = Bz$ and ↓ ↙

"how do you do a coord. transform on a matrix?"

Similarity transform on A.

$x' = Ax$ $\quad x' = Pz'$, $x = Pz$

plugging in $\quad Pz' = APz \Rightarrow z' = \underbrace{P^{-1}AP}_{B} z$

B is related to A by a similarity transform

A & B have the same eigen values

same determinant, same trace ...

different eigen vectors.

$$x' = Ax \Rightarrow z' = \underbrace{P^{-1}}_{\text{transform back to } z}\underbrace{A}_{\text{apply } A}\underbrace{P}_{\text{transform to } x}\underbrace{z}_{z \text{ coords}}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvectors:

v is a right-eigenvector of A if $\lambda v = Av$

w^T is a left-eigenvector of A if $\lambda w^T = w^T A$

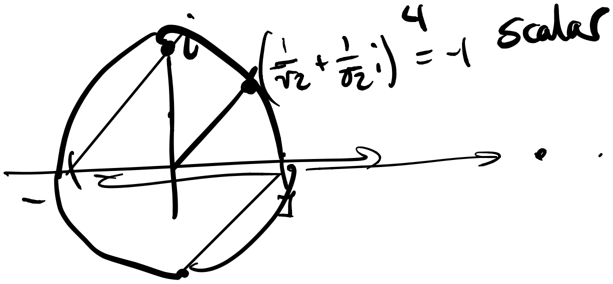
} λ is the eigen value scalar

λ is a eigenvalue

TANGENT:

$\det(\lambda I - A) = 0$

$\sqrt{b^2 - 4ac}$



$\sqrt{x^T x}$ $\quad z = a+bi \xrightarrow[\text{mag}]{} \quad \sqrt{a^2+b^2} = \sqrt{z^* z} = \sqrt{(a-bi)(a+bi)}$

$a^2 \cancel{-bia+bia}$

$+b^2 \cancel{i^2}$

λ can be real or complex depending on whether or not A stretches and/or rotates vectors

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$\boxed{\lambda v = Av}$, $\boxed{\lambda w^T = w^T A}$

Suppose we can find a basis of right eigenvectors...

$P = [v_1 \cdots v_n]$ $\qquad x' = Ax \qquad x = Pz$

$$AP = [Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n] = \underbrace{[v_1 \cdots v_n]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{:= D}$$

EIGENVECTOR EQN:

$$\boxed{AP = PD} \Rightarrow \boxed{A = PDP^{-1}}$$

diagonalization of A

A is related to D by similarity transform.

cols of P are right evecs.

rows of P^{-1} are left evecs $\rightarrow$ why ?

$$AP = PD \rightarrow \boxed{P^{-1}A = DP^{-1}}$$

$$\begin{bmatrix} -w_1^T- \\ -w_n^T- \end{bmatrix} A = \boxed{\begin{bmatrix} w_1^T A \\ \vdots \\ w_n^T A \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix} \begin{bmatrix} -w_1^T- \\ \vdots \\ -w_n^T- \end{bmatrix}}$$

insummary: right eigen vectors ↓ eigen values ↓ left eigen vectors ↓

$$A = \underbrace{\begin{bmatrix} v_1 \cdots v_n \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} - w_1^T - \\ - w_n^T - \end{bmatrix}}_{P^{-1}}$$

A is "diagonalizable"

$$= \sum_i \lambda_i \underbrace{v_i w_i^T}_{\text{outer product}}$$

Note: not always possible, but pick a random A it will be

if its not true: you have to look for solutions to $v + \lambda u = Au$

when generalized eigenvectors

you can always a basis of these guys

{ generalized eigen vectors

D is not diagonal anymore instead it has the form

$$J = \begin{bmatrix} \begin{matrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{matrix} & 0 & 0 \\ 0 & \boxed{\begin{matrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{matrix}} & 0 \\ & & \ddots \end{bmatrix} \Rightarrow A = PJP^{-1}$$

Jordan blocks

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}$$

$$A^n = PD\cancel{P^{-1}}PD\cancel{P^{-1}}\cdots PDP^{-1} = P\underbrace{D^n}P^{-1}$$

$$\alpha A^k + \alpha_2 A^{k-1} \qquad = \qquad \begin{bmatrix} \lambda_1^n & & 0 \\ 0 & \ddots & \lambda_n^n \end{bmatrix}$$

$$\alpha PD^kP^{-1} + \alpha_2 PD^{k-1}P^{-1} + \cdots$$

$$P(\alpha D^k + \alpha_2 D^{k-1} + \cdots)P^{-1}$$

polynomial function $f(A) = Pf(D)P^{-1} = P\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}P^{-1}$

SPECTRAL MAPPING THM:

want to compute e^{At} matrix exponential

critical for solutions to $\dot{x} = Ax$, $x(0) = x_0$

polynomial func of A $\quad x(t) = e^{At}x(0)$

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$