Tuesday, November 26, 2019 7:09 PM

$\Delta J = \frac{\partial J}{\partial u}\Delta u$

$\Rightarrow \Delta J = \frac{\partial J}{\partial x[T]}\frac{\partial x[T]}{\partial u}\Delta u$

$\left[\frac{\partial l}{\partial x}\Big|_T \; 1\right]\left[A_{T-1}\cdots A_1 B_0 \,\middle|\, \cdots \,\middle|\, A_{T-1}B_{T-2} \,\middle|\, B_{T-1}\right]\begin{bmatrix}\Delta u[0]\\ \vdots \\ \Delta u[T-1]\end{bmatrix} = \left[\lambda[1]B_0 \;\; \lambda[2]B_1 \cdots \lambda[T-1]B_{T-2} \;\; \lambda[T]B_{T-1}\right]\begin{bmatrix}\Delta u[0]\\ \vdots \\ \Delta u[T-1]\end{bmatrix}$

n: # states

m: # inputs

$\frac{\partial J}{\partial u[t]} = \underbrace{\left[\frac{\partial l}{\partial x}\Big|_T \; 1\right]}_{\text{row vec } 1\times(n+1)} \underbrace{A_{T-1}\cdots A_{t+1}}_{\text{matrix}} B_t$

$\left[\frac{\partial J}{\partial u[0]} \cdots \frac{\partial J}{\partial u[T-2]} \; \frac{\partial J}{\partial u[T-1]}\right]$

$\left[\frac{\partial l}{\partial x}\Big|_T \; 1\right]\underbrace{\begin{bmatrix}\frac{\partial f}{\partial x}\big|_{T-1} & 0\\ \frac{\partial l}{\partial x}\big|_{T-1} & 1\end{bmatrix}}_{A_{T-1}}\underbrace{\begin{bmatrix}\frac{\partial f}{\partial x}\big|_{T-2} & 0\\ \frac{\partial l}{\partial x}\big|_{T-2} & 1\end{bmatrix}}_{A_{T-2}}\cdots$ } compute these terms recursively

$\left[\frac{\partial l}{\partial x}\Big|_T \frac{\partial f}{\partial x}\Big|_{T-1} + \frac{\partial l}{\partial x}\Big|_{T-1} \;\; 1\right]\begin{bmatrix}\frac{\partial f}{\partial x}\big|_{T-2} & 0\\ \frac{\partial l}{\partial x}\big|_{T-2} & 1\end{bmatrix}$ $\quad \lambda^T[T-2]$

$\left[\frac{\partial l}{\partial x}\Big|_T \frac{\partial f}{\partial x}\Big|_{T-1}\frac{\partial f}{\partial x}\Big|_{T-2} + \frac{\partial l}{\partial x}\Big|_{T-1}\frac{\partial f}{\partial x}\Big|_{T-2} + \frac{\partial l}{\partial x}\Big|_{T-2} \;\; 1\right]$

unaugmented the sys.

$$\boxed{\lambda^T[t-1] = \lambda[t]^T\frac{\partial f}{\partial x}\Big|_{t-1} + \frac{\partial l}{\partial x}\Big|_{t-1}, \quad \lambda[T] = \frac{\partial l}{\partial x[T]}}$$

from $\lambda[t+1]$: $\frac{\partial J}{\partial u[t]} = \lambda[t+1]^T\frac{\partial f}{\partial u}\Big|_t + \frac{\partial l}{\partial u}\Big|_t$

solve: $\lambda^T[t-1] = \lambda[t]^T\frac{\partial f}{\partial x}\Big|_{t-1} + \frac{\partial l}{\partial x}\Big|_{t-1}$ $\quad \lambda[T]^T = \frac{\partial l}{\partial x[T]}$

$\frac{\partial J}{\partial u[t]} = \lambda[t+1]^T\frac{\partial f}{\partial u}\Big|_t + \frac{\partial l}{\partial u}\Big|_t$

Computer function for $\frac{\partial J}{\partial u}$:

full control vector... (open loop) → solver

function [out] = dJdu(u, x_0)

→ compute $x(t)$ by plugging $u(t)$ into $x[t+1] = f(x,u,t)$, $x[0] = x_0$

→ linearize around $x(t)$, $u(t)$

→ solve for costate: $\lambda^T[t-1] = \lambda[t]^T\frac{\partial f}{\partial x}\Big|_{t-1} + \frac{\partial l}{\partial x}\Big|_{t-1}$ $\quad \lambda[T]^T = \frac{\partial l}{\partial x[T]}$ backwards recursion

→ solve for gradient: $\frac{\partial J}{\partial u[t]} = \lambda[t+1]^T\frac{\partial f}{\partial u}\Big|_t + \frac{\partial l}{\partial u}\Big|_t$

out = $\left[\frac{\partial J}{\partial u[0]} \cdots \frac{\partial J}{\partial u[T-1]}\right]$

Gradient Descent Code

terrible pseudo code

u = [ ] ← initial control guess

solver

for k = 1:K
  dJdu = [0 ... 0]
  for t = 1:T-1
    $x[t+1] = f(x,u,t)$ $\quad x[0] = x_0$
  for t = T:-1:2
    $\lambda^T[t-1] = \lambda[t]^T\frac{\partial f}{\partial x}\Big|_{t-1} + \frac{\partial l}{\partial x}\Big|_{t-1}$ $\quad \lambda[T] = \frac{\partial l}{\partial x}\Big|_T$
  dJ du[t] = $\lambda[t+1]^T\frac{\partial f}{\partial u}\Big|_t + \frac{\partial l}{\partial u}\Big|_t$
u = u − α dJdu
[ ] − α [ ]

Functions:
$f(x,u,t)$
$l(x,u,t)$
$\frac{\partial f}{\partial x}$
$\frac{\partial l}{\partial x}$
$\frac{\partial f}{\partial u}$
$\frac{\partial l}{\partial u}$

repeat

$J(u) = \sum_{t=0}^{T-1}\left(l(x,u,t)\right) + l(x[T])$

$l(x,u,t)$ in LQR is $x(t)^T Q(t) x(t) + u(t)^T R(t) u(t)$

pick

need $x(t)$ from $u(t)$

pseudo code J(u)...

$u(t) \to x(t)$ $\quad x[t+1] = f(x,u,t)$ $\quad x[0] = x_0$

$J = \sum_{t=0}^{T-1}\left(l(x,u,t)\right) + l(x[T])$

Continuous time version: costate propagation for dynamics

$t=0$

Continuous time version: Costate propagation

for dynamics
$\dot{x} = f(x,u,t)$
$J = \int_0^T \ell(x,u,t)\,dt + \ell(x(T))$

$-\dot{\lambda}(t) = \lambda^T \frac{\partial f}{\partial x}\Big|_t + \frac{\partial \ell}{\partial x}\Big|_t \qquad \lambda(T) = \frac{\partial \ell}{\partial x}\Big|_T \qquad \frac{\partial J}{\partial u(t)} = \lambda(t)^T \frac{\partial f}{\partial u}\Big|_t + \frac{\partial \ell}{\partial u}\Big|_t$

LQR

$-\dot{\lambda} = \lambda^T A + x^T Q \qquad \lambda(T) = x(T)^T Q_T \qquad \boxed{\frac{\partial J}{\partial u(t)}} = \lambda^T B + u^T R = 0$ ← not doing anymore

$u = R^{-1} B^T \lambda$

Examples:

in time interval $[0,T]$
make a ball role as far as possible

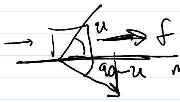
related to...
Brachistochrone

$$\max_{u(t)} \; x[T] - (y[T] - \bar{y})^2$$

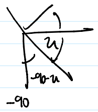
s.t. $\dot{x} = v\cos(u) \qquad x(0) = 0$
$\dot{y} = v\sin(u) \qquad y(0) = 0$
$\dot{v} = -g\sin(u) \qquad v(0) = 0$
a

target $y[T] = \bar{y}$

$u(t) = 0$



$ma = f = mg\cos(90-u)$



$a = g\cos(-90-u)$
$= g\cos(90+u)$
$- \sin(u)$

Example:

$$\min \; (\bar{\theta} - \theta(T))^2 + \dot{\theta}(T)^2$$

$\bar{\theta}$ = pendulum upright

s.t. inverted pendulum

$\dot{\theta}$

H_2 & H_∞ Control

$G(s)$: matrix of transfer functions

TF: $V(s) \to \boxed{G(s)} \to Z(s)$

ways to measure the "size" or energy gain of system

H_2 norm:

$$\|G(s)\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\mathrm{Tr}\left(G(j\omega)^* G(j\omega)\right)}_{\|G(j\omega)\|_F^2} d\omega\right)^{1/2}$$

integral

vectorizing $G(j\omega)$, and then taking the 2-norm

$$\|G(j\omega)\|_F^2 = \left(\sum_{ij} G_{ij}(s)^2\right)$$

$$\sum_\omega \sum_{ij} (G_{ij})^2$$ average or total energy gain of output

RMS energy of the input

H_∞ norm:

$$\|G(s)\|_\infty = \max_\omega \bar{\sigma}(G(j\omega))$$

$$\max_\omega \max_{V\neq 0} \frac{\|G(j\omega)V\|_2}{\|V\|_2}$$

$$\frac{\sqrt{V^T G^T G V}}{\sqrt{V^T V}}$$

$$\max_\omega \max_V \frac{\|GV\|_2}{\|V\|_2}$$ worst case energy gain of output

worst direction

RMS energy of output for the worst case input of mag 1.

$$\int_{-\infty}^{\infty} F^*(j\omega)\, G(j\omega)\, d\omega =$$
$$\int_{-\infty}^{\infty} f(t)^* g(t)\, dt$$

RMS energy of the output when the input white noise of mag 1

Parseval's Thm

energy of case output for of mag 1.

$\int_{-\infty}^{\infty} f(t)^T g(t)\, dt$

$F(j\omega)$ is Fourier transform of $f(t)$

Time Domain:

$$\|G(s)\|_2 = \left(\int_0^\infty \mathrm{Tr}\left(g(t)^T g(t)\right) dt\right)^{1/2}$$

$g(t)$: matrix of impulse responses

$g_{ij}(t)$: impulse response of $G_{ij}(s)$

$$\|G(s)\|_\infty = \max_\omega \max_{v(t)} \frac{\int_0^\infty z(t)^T z(t)\, dt}{\int_0^\infty v(t)^T v(t)\, dt}$$

where $v(t)$ is a sinusoid with freq. ω

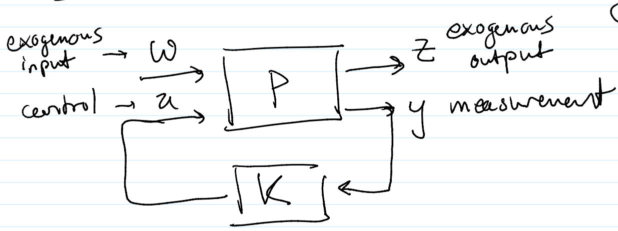
$\min \|G(s)\|_2$ easy...

$\min \|G(s)\|_\infty = \min \max_\omega \max_V \frac{\|Gv\|_2}{\|V\|_2}$ hard to solve

| either... worst case $\leftarrow$ robustness

but average performance

General Control Formulation:



goal: $z = \boxed{G_{zw}(P,K)}\, w$

$\min_K \|G_{zw}(P,K)\|_2$ or $\min_K \|G_{zw}(P,K)\|_\infty$

pick w & z to define performance of your system...

state space model:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \rightarrow$$

Special case of H_2 optimization: LQG

noise model
white noise

$$\dot{x} = Ax + Bu + w_d$$
$$y = Cx + w_n$$

$$E\left(\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \begin{bmatrix} w_d(\tau) & w_n(\tau) \end{bmatrix}\right) = \begin{bmatrix} W_d & 0 \\ 0 & W_n \end{bmatrix} \delta(t-\tau)$$

Find $u = K(s)\, y$

Find $u = K(s)\, y$

$$J = E\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T [x^TQx + u^TRu]\, dt\right] \quad \text{with} \quad \begin{matrix} Q = Q^T \geq 0 \\ R = R^T \geq 0 \end{matrix}$$

$$z = \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \qquad \begin{pmatrix} w_d \\ w_n \end{pmatrix} = \begin{bmatrix} W_d^{1/2} & 0 \\ 0 & W_n^{1/2} \end{bmatrix} w$$

← white noise w unit norm

$$J = E\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T z(t)^T z(t)\right] = \| G_{zw}(P,K)\|_2$$

with

$$\dot{x} = Ax + [W_d^{1/2} \;\; 0]\, w + Bu$$

$$z = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u$$

$$y = Cx + [0 \;\; W_n^{1/2}]\, w$$

state space model s.t. LQG is $\min_K \| G_{zw}(P,K)\|_2$

→ $z^Tz = x^T Q^{T/2} Q^{1/2} x + u^T R^{T/2} R^{1/2} u$

Solving $\min_K \| G_{zw}(P,K)\|_\infty$ hard

look for controllers $\| G_{zw}(P,K)\|_\infty < \gamma$, search over γ

$$\min_K \max_w \int_0^\infty \underbrace{z(t)^T z(t)}_{\text{magnitude of output}} - \gamma^2 \underbrace{w(t)^T w(t)}_{\text{mag. input}}\, dt$$

Competition between controller and input.

searching for a small enough gamma where the controller can win.

$[\gamma_-, \gamma_+] \leftarrow [0, \bar{\gamma}]$

- set $\gamma = \dfrac{\gamma_+ + \gamma_-}{2}$
- try to find a controller
  if find one, set $\gamma_+ = \gamma$

- try to find a controller
  - ↳ if you can find one, set $\gamma_+ = \gamma$
  - ↳ if not set $\gamma_- = \gamma$
- repeat to find smallest γ.