Eigenvalues, Eigenvectors, Diagonalization

$A \in \mathbb{R}^{n\times n}$ diagonalizable

$$A = PDP^{-1} = \underbrace{\overset{P}{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}}}_{\text{right eigen vectors}} \underbrace{\overset{D}{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}}_{\text{eigenvals on diagonal}} \underbrace{\overset{P^{-1}}{\begin{bmatrix} -w_1- \\ \vdots \\ -w_n- \end{bmatrix}}}_{\text{left eigen vectors}}$$

~ spectrum of A

Right eigenvectors: $\lambda_i v_i = Av_i$

left eigenvectors: $\lambda w_i^T = w_i^T A$

$$= \sum_i \lambda_i \underbrace{v_i w_i^T}_{\text{outer product rank 1 "dyad"}}$$

Notes: $w_i^T v_j = 0$ $i \neq j$

$w_i^T v_i = 1$ } $\rightarrow$ $P^{-1}P = I$

subtlety...

$$A = PDP^{-1} = P\underset{\text{diagonal}}{E}D\underset{}{E^{-1}}P^{-1}$$

"if you scale up the length of right evecs → scale down length of left evecs in diagonalization"

reason: simplify action of A

"to find coordinates in which A just stretches vector components"

Computing powers of A:

$$A^k = \underbrace{PD\underbrace{P^{-1} \times P}_{I}DP^{-1} \times \cdots \times PDP^{-1}} = PD^nP^{-1} = P\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_A^k \end{bmatrix}P^{-1}$$

$\Rightarrow f(A)$: f is ${}^{x\ K}$ a polynomial

$$f(A) = Pf(D)P^{-1} = P\begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}P^{-1}$$

spectral mapping theorem

$$A^{-1} = PD^{-1}P^{-1}$$

A^{-1}: has the same left & right eigenvectors

eigenvalues = eigenvalues of A inverted.

$$AA^{-1} = I$$

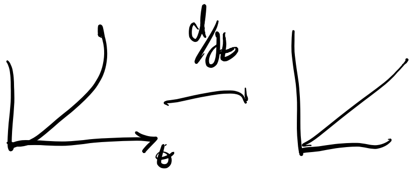
$$\underbrace{\underbrace{PD\underbrace{P^{-1}P}_{I}D^{-1}P^{-1}}_{I}}_{I} = I$$

e^A: important for control systems

Exponentials:

e^t is the function such that $\frac{d}{dt}e^t = e^t$ this is how e is defined.

differentiating polynomials



power drops 1... $\frac{d}{dt}\cancel{t}^k = kt^{k-1}$

power series defn of e :

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + = \sum_k \frac{t^k}{k!!}$$

diff ... $\downarrow$ 0 $\quad\downarrow$ 1 $\quad\downarrow$ t $\quad\downarrow$ $\frac{1}{2!}t^2$ +

applying chain rule...

side note.

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t} \implies f(t) = e^{\lambda t} \Big\} \text{ are eigenfunctions of } \frac{d}{dt} \text{ operator}$$

Matrix Exponential:

defn: $e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots = \sum_k \frac{A^k}{k!}$

So $\frac{d}{dt}e^{At} = Ae^{At}$

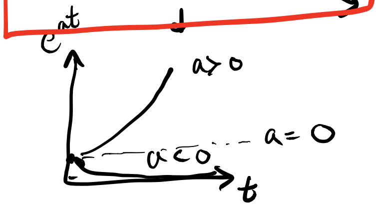
$$A = PDP^{-1} \implies e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t}\end{bmatrix}P^{-1}$$

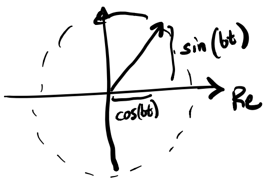
$\lambda_1 \ldots \lambda_n$ can be real or complex

so suppose $\lambda = a+bi \Rightarrow e^{\lambda t} = e^{(a+bi)t}$

$= e^{at}e^{bit}$

$$\boxed{e^{\lambda t} = \overset{\text{real}}{e^{at}}\, e^{bit}} = e^{at}(\cos(bt) + i\sin(bt))$$

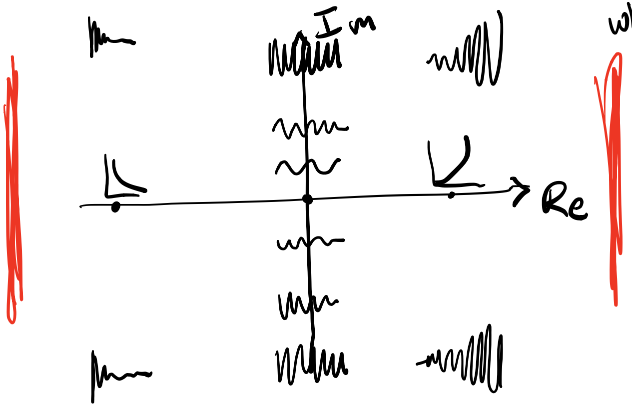




as t increase this phasor spins

b rate of spinning, "frequency"

Eigenvalues in Complex Plane



what does $e^{\lambda t} = e^{\overset{\text{amplitude}}{at}} e^{bti}$ → freq.
do for λ in different parts of complex plane

Finding eigenvalues:

find λ s.t. $\lambda v = Av$ for some v

$$(\lambda I - A)\underset{\sim}{v} = 0$$

v is in the null space of $\lambda I - A$ } needs to have nontrivial null space $\Longleftrightarrow$ $\boxed{\det(\lambda I - A) = 0}$

from diagonalization

$$\lambda I - A = \lambda PP^{-1} - PDP^{-1} = P(\lambda I - D)P^{-1}$$

$$= P \begin{bmatrix} \lambda - \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda - \lambda_n \end{bmatrix} P^{-1}$$

$\chi(A)$ ~ order n $s^n + \alpha_{n-1}s^{n-1} + \dots$

$$\overbrace{\det(\lambda I - A)}^{\chi(A)} = 0$$

characteristic polynomial of A → eigenvalues are roots of char poly.

once you find $\lambda_i \dots$ s.t. $\det(\lambda_i I - A) = 0$ find v_i (or w_i) by computing nullspace of $\lambda_i I - A$

Cayley Hamilton Thm: (useful for controllability)

$$\underline{\chi(A)} = 0 \qquad \chi(A) = P\chi(D)P^{-1}$$

$$= P\begin{bmatrix}\chi(\lambda_1)=0 & & \\ & \ddots & \\ & & \chi(\lambda_n)=0\end{bmatrix}P^{-1}$$

$$\chi(A) = A^n \underbrace{+ \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + A_0}_{} = 0 \quad \Leftarrow$$

this implies that A^n can always be written as as sum of A^{n-1} & lower order terms

$$A^n = -\alpha_{n-1}A^{n-1} + \cdots - \alpha_1 A - A_0$$

So nxn matrix polynomials of any order can be written with order n−1.

Preview: $\underline{e^{At}}Bu(t)$ $\quad [A^{\infty}B \cdots | \underline{A^{n-1}B \cdots A^3B \; A^2B \; AB \; B}]$

more later.

Differential Eqns:

simplest case: $\dot{x}(t) = \underset{\sim}{A}\,\underline{x(t)}$ $\qquad \underline{x(0) = x_0}$

~~state transition matrix~~

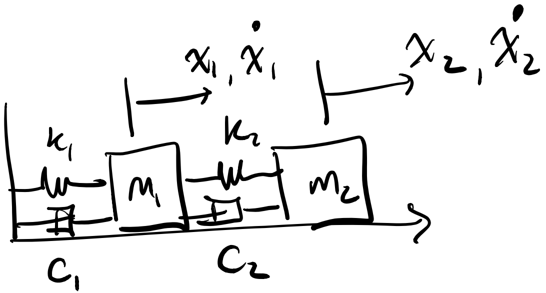
state vector

initial state or initial condition

linear differential eqn → Linear time invariant (LTI)

Examples of states:

position, velocity



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix} \begin{matrix}0&0\\0&0\end{matrix} & \overset{I}{\boxed{\begin{matrix}1&0\\0&1\end{matrix}}} \\ \underset{\text{spring physics}}{\boxed{k/m}} & \underset{\text{damper physics}}{\boxed{c/m}} \end{bmatrix} \underbrace{\begin{bmatrix}x_1\\ x_2\\ \dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{x}$$

force spring 1: $= k_1 x_1$

force spring 2: $= k_2(x_1 - x_2)$

$$\boxed{\longleftarrow \quad f = ma}$$

$a = \dfrac{f}{m}$

force damper 1 $= c_1(\dot{x}_1)$

" " 2 $= c_2(\dot{x}_1 - \dot{x}_2)$

RLC: analogs

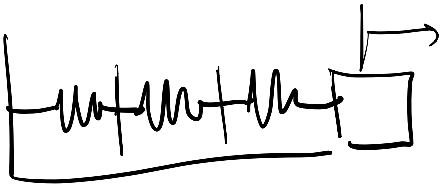
springs: capacitors

dampers: resistors

masses: inductors

} → physics equations KVL, KCL,

2nd order system



Solve $\dot{x} = Ax$ $\quad x(0) = x_0$ $\quad \Rightarrow \quad \underbrace{x(t)}_{\text{state vector at time } t} = \underbrace{e^{At}}_{\substack{\downarrow \\ \text{state transition matrix}}} \underbrace{x_0}_{\text{initial state}}$

$$\dot{x}(t) = A\underbrace{e^{At}x_0}_{x(t)} = Ax(t)$$

why we care about evals ...

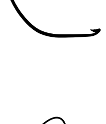
$$x(t) = e^{At}x_0 = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} P^{-1}x_0$$

$$\underbrace{P^{-1}x(t)} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_n t} \end{bmatrix} \underbrace{P^{-1}x_0}_{z_0} = e^{Dt}z_0$$

$$z(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} (z_0)_1 \\ \vdots \\ (z_0)_n \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}(z_0)_1 \\ \vdots \\ e^{\lambda_n t}(z_0)_n \end{bmatrix}$$

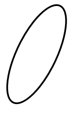
eigenvectors v_i each evolve separately according to λ_i

Eigenvectors or "eigenmodes" or "modes": are $\boxed{e^{at}}$ amplitude



Eigen mode v_k is

stable if $e^{\lambda_k t}(z_0)_k \to 0$ as $t \to \infty$ "decaying" $Re(\lambda_k) < 0$



unstable if $e^{\lambda_k t}(z_0)_k \to \infty$ as $t \to \infty$ "blowing up" $Re(\lambda_k) > 0$

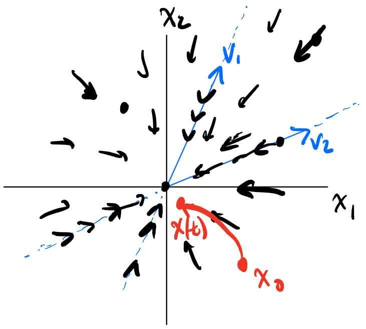
if $e^{\lambda_k t}(z_0)_k$ bounded $t \to \infty$ $\quad Re(\lambda_k) = 0$

Pictures: 2-dimensional $A \in \mathbb{R}^{2\times2}$ $A = PDP^{-1}$

"vector field" ← $\boxed{\dot{x} = Ax}$ $[v_1\ v_2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}[v_1\ v_2]^{-1}$

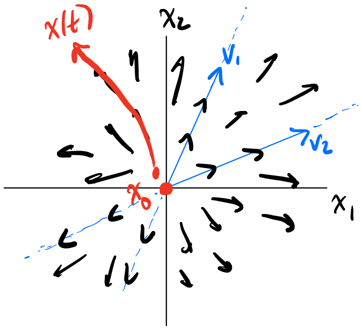
arrows

point in space



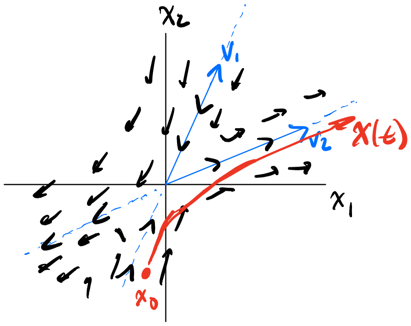
λ_1, λ_2 real $\lambda_1, \lambda_2 < 0$

STABLE SYSTEM



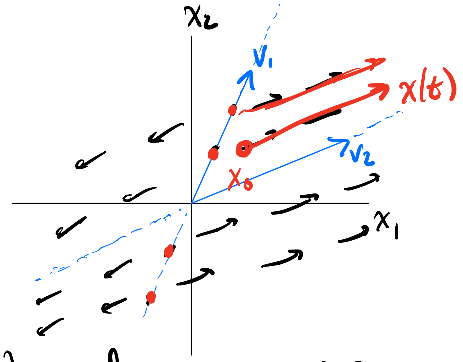
λ_1, λ_2 real $\lambda_1, \lambda_2 > 0$

UNSTABLE



λ_1, λ_2 real $\lambda_1 < 0 < \lambda_2$

SADDLE



λ_1, λ_2 real $\lambda_1 = 0, \lambda_2 > 0$

Note: arrows closer to the origin should be smaller cause $|\dot{x}|$ scales with $|x|$

Real λ ↑

Complex eigenvalues:

found λ_k as roots of $\det(\lambda I - A) = 0$ can be complex

real matrices can still have complex roots

$\lambda^2 + 2\lambda + 1 = 0$

$\lambda^2 + 1 = 0$ → complex roots

For real A:

- complex eigenvalues come in conjugate pairs
- complex eigenvectors come in conjugate pairs

$\lambda_1 = a + bi$ $\quad b \neq 0$ $\qquad \lambda_2 = a - bi$

$\downarrow$ eigenvector $\qquad\qquad \downarrow$

$r_1 = u - vi$ $\qquad r_2 = u + vi$ $\quad$ right eigenvectors

u, v: real $\quad$ real

left eigenvectors $\quad r_1^{*} = [- u^T + v^T i -]$ $\quad$ conjugate transpose

$l_1^{*} = w^T + y^T i$

$l_2^{*} = w^T - y^T i$

w, y: real

$$A = \begin{bmatrix} | & | & \\ r_1 & r_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} a+bi & 0 & 0 \\ 0 & a-bi & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} - & l_1^{*} & - \\ - & l_2^{*} & - \\ & & \end{bmatrix} \qquad \begin{bmatrix} - & l_1^{*} & - \\ - & l_2^{*} & - \end{bmatrix} \begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix} = I$$

$$= \boxed{\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}(u-vi) & (u+vi)\frac{1}{\sqrt{2}} \\ | & | \end{bmatrix}} \cdots \begin{bmatrix} \boxed{\begin{matrix} a+bi & 0 \\ 0 & a-bi \end{matrix}} & \\ & \ddots \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} \frac{1}{\sqrt{2}}(w^T + y^T i) - \\ \frac{-1}{\sqrt{2}}(w^T - y^T i) - \end{matrix}} \\ \\ \end{bmatrix}$$

just look at 2d subspace...

$$\boxed{\begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}}_{U}} \boxed{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}}_{U^{*}} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \frac{1}{\sqrt{2}}} \boxed{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix}}$$

$U U^{*} = I \qquad I$

Real a $\quad \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ $\qquad$ imag b $\quad \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$

mag

stretching $\qquad\qquad\qquad$ rotation

$\downarrow$

$UU^* = I$

$$\begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix}$$

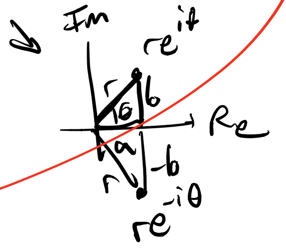
general form instead of diagonalization for real matrices w complex eigenvalues

$$A = \begin{bmatrix} | & | & \\ u & v & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} a & -b \\ b & a \end{matrix}} & \\ & \lambda_3 \ddots \end{bmatrix} \begin{bmatrix} - & w^T & - \\ - & y^T & - \\ & & \end{bmatrix}$$

$r = \sqrt{a^2+b^2}$ ✓

$$\begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix} \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} = I$$

$$\lambda_{1,2} = a \pm bi = re^{\pm i\theta}$$



?

$$r \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$ 2x2 rotation matrix

stretching b $r = \sqrt{a^2+b^2}$

rotating within the eigenspace defined by u & v

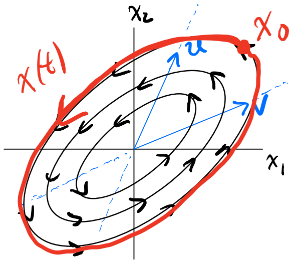
To summarize:

$$\dot{x} = Ax \qquad A = \frac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ u-vi & u+vi \\ | & | \end{bmatrix} \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix} \begin{bmatrix} | & | \\ u-vi & u+vi \\ | & | \end{bmatrix}^{-1}$$

$$x(t) = e^{At}x(0) \qquad e^{At} = \frac{1}{\sqrt{2}} \begin{bmatrix} | & | \\ u-vi & u+vi \\ | & | \end{bmatrix} \begin{bmatrix} e^{at}e^{bti} & 0 \\ 0 & e^{at}e^{-bti} \end{bmatrix} \begin{bmatrix} | & | \\ u-vi & u+vi \\ | & | \end{bmatrix}^{-1} \sqrt{2}$$

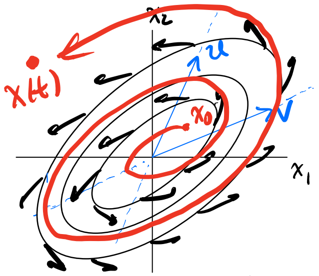
$$= \underbrace{\begin{bmatrix} u & v \end{bmatrix}}_{\text{shape of ellipse}} \underbrace{e^{at}}_{\text{Scaling by } e^{at}} \underbrace{\begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}}_{\text{rotation by } \underset{\text{total rotation}}{\underline{bt}}} \underbrace{\begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix}^{-1}}_{\begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix}}$$

b rotation speed



$\lambda_1, \lambda_2 = a \pm bi$

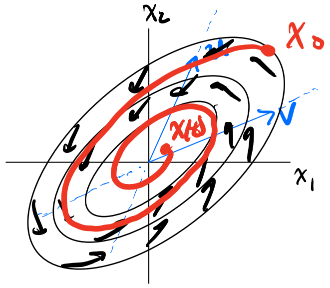
$a = 0 \Rightarrow e^{at} = 1$



$\lambda_1, \lambda_2 = a \pm bi$

$a > 0 \Rightarrow e^{at}$ blow up

complex unstable



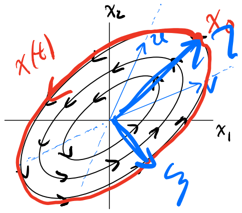
$\lambda_1, \lambda_2 = a \pm bi$

$a < 0 \Rightarrow e^{at}$ decays

Complex sta

$$A = PEDE^{-1}P^{-1}$$

$$= \underline{\begin{bmatrix} u & v \end{bmatrix} R} \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \underset{R^T}{\underline{R^{-1}}} \begin{bmatrix} - & w^T & - \\ - & y^T & - \end{bmatrix}$$



$\lambda_1, \lambda_2 = a \pm bi$

$a = 0 \Rightarrow e^{at} = 1$

$$\frac{\begin{bmatrix} u & v \end{bmatrix} R}{\begin{bmatrix} \eta & \xi \end{bmatrix}}$$

Dynamical Systems w control inputs:

Before $\dot{x}(t) = A x(t)$

Now add control input $u(t)$

$$\underbrace{\dot{x}}_{\dot{x} \in \mathbb{R}^n}(t) = \underbrace{A}_{A \in \mathbb{R}^{n \times n}} \underbrace{x(t)}_{x \in \mathbb{R}^n} + \underbrace{B}_{B \in \mathbb{R}^{n \times m}} \underbrace{u(t)}_{u \in \mathbb{R}^{\textcircled{m}}}$$

$\textcircled{m} \rightarrow$ number of inputs you have

$$Bu = \sum_k B_k u_k$$

$$B = \begin{bmatrix} | & & | \\ B_1 & \cdots & B_m \\ | & & | \end{bmatrix}$$

what is the soln ...

before $x(t) = e^{At} x(0)$

now $$x(t) = \underbrace{\underbrace{e^{At}}_{\text{how i.c. evolves}} \underbrace{x(0)}_{\text{initial condition}}}_{\text{drift term}} + \int_0^t \underbrace{e^{A(t-\tau)}}_{\text{how an input at time } \tau \text{ evolves up to the present } t} \underbrace{Bu(\tau)}_{\text{input to system at time } \tau} d\tau$$

$\int_0^t \rightarrow$ summing over all input times τ between time 0 & present time t

Discrete time: $e^{A\Delta t}$ (with the A from before)

$$x[t+1] = \tilde{A}\, x[t]$$ update eqn (as opposed to differential eq)

by abuse of notation

A in discrete time is actually $e^{A\Delta t}$ (from the A in continuous time)

different stability conditions:

for continuous time A: $Re(\lambda) < 0$

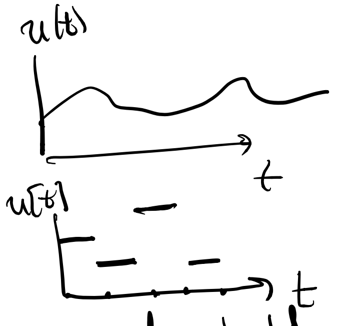
for discrete time A: $|\lambda| < 1$ $\quad \lambda = e^{\lambda \Delta t} = e^{a\Delta t} e^{ib\Delta t}$

(again abusing notation) $\to < 1$

Adding a control input:

$$x[t+1] = Ax[t] + \overset{\uparrow}{B}u[t]$$

~~same~~ B



zero order hold

$$B = \int_0^{\Delta t} e^{A(\Delta t - \tau)} B \, d\tau$$

$\downarrow$ B from before

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A[Ax[0] + Bu[0]] + Bu[1]$$

$$\vdots$$

$$x[t] = A^t x[0] + \sum_{\tau=0}^{t-1} A^{t-\tau-1} B u[\tau]$$ $\to$ compare w continuous time form

$$x[t] = A^t x[0] + \sum_{\tau=0}^{t-1} \underline{A^{t-\tau-1}}\, \underline{Bu[\tau]}$$

$t-(t-1)-1$

$0 + t - 1$

↓ evolution of init state.

↓ init. state

↓ sum over all inputs

↓ evolution of input from τ to t

input at τ

$$x[t] = A^t \underline{x[0]} + \underbrace{\left[A^{t-1}B \middle| \cdots A^2B \middle| AB \middle| B\right]} \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[t-1] \end{bmatrix}$$

use to test for controllability

↓

$$\underline{x[t] - A^t x[0]} = \underline{\left[A^{t-1}B \middle| \cdots \middle| AB \middle| B\right]} \begin{bmatrix} u[0] \\ \vdots \\ u[t-1] \end{bmatrix}$$

want to pick

Columns of this matrix need to span $\mathbb{R}^n$

Measurement Equation

$C \in \mathbb{R}^{p \times n}$ → fat

$$y(t) = Cx(t) + Du(t)$$

$$y[t] = Cx[t] + Du[t]$$

Matrix Facts:

Rotation Matrix: R : $\underbrace{R^T R} = I$ det(R)=1

$x = Rz$ → doesn't change angles & lengths (inner product)

picture: orthonormal coordinate sys

Symmetric Matrix: $Q = Q^T$

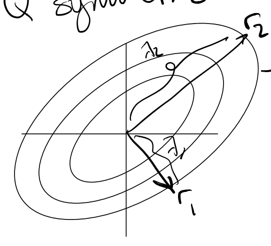
Q : only real evals

: orthonormal eigenvectors

: diagonalize $Q = RD\underset{R^{-1}}{\underline{R^T}}$

picture: $\underbrace{x^T Q x}_{\text{level sets}}$: quadratic form

Q symmetric $x^T Q x = \text{const.}$



→ $x^T Q x = \text{const}$

$$Q = RDR^T$$
$$= \begin{bmatrix} r_1 & r_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix}$$

if $\lambda_1, \lambda_2 > 0$ Q is positive definite ...

$\Rightarrow x^T Q x > 0$ for any x.

Q is positive semi definite

$\Rightarrow x^T Q x \geq 0$ for any x