OUTLINE:

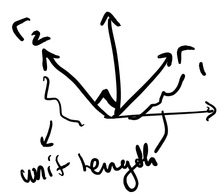
- LIN ALG.
- LTI SYSTEMS
- CONTROLLABILITY / OBSERVABILITY
- FEEDBACK CONTROL / OBSERVER DESIGN

$\left.\begin{matrix}\text{CONTROLLABILITY / OBSERVABILITY} \\ \text{FEEDBACK CONTROL / OBSERVER DESIGN}\end{matrix}\right\} \rightarrow$ DUAL PROBLEMS

LIN ALG.

SPECIAL MATRICES

- ROTATION MATRICES: $R \in \mathbb{R}^{n \times n}$ $\quad R^T R = I, \ \det(R) = 1.$



$R = \begin{bmatrix} | & | \\ r_1 & r_2 \\ | & | \end{bmatrix}$ Rotation Matrices
- orthonormal coordinate systems
- preserve metric properties

$x = Rz \quad x^T x = z^T R^T R z = z^T z$

- Symmetric Matrices:

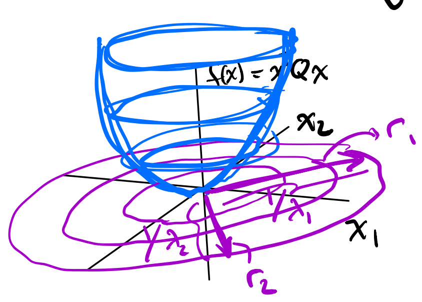
$Q \in \mathbb{R}^{n \times n} \quad Q = Q^T$

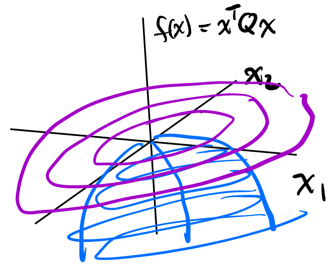
- have pure real eigenvalues
- diagonalized by a rotation matrix

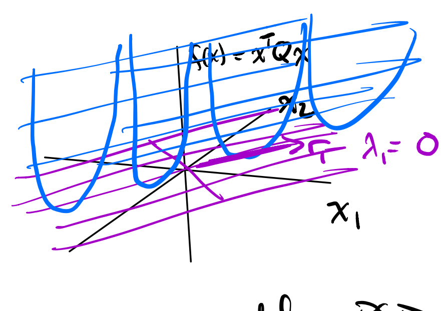
$Q = R \underbrace{D}_{\text{real vals on diag}} R^T$

Quadratic Form:

$f(x) = x^T Q x \in \mathbb{R}$







Q pos def. or PD

$x^T Q x > 0 \quad \forall x \rightarrow \lambda_1 \ldots \lambda_n > 0$

$x^T R D \underbrace{R^T x}_{\text{rotated coord sys}} \quad R = [r_1 \ r_2]$

$r_1^T [r_1 \ r_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} r_1$

$[1 \ 0] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1$

Q neg def or ND

$x^T Q x < 0 \quad \forall x \rightarrow \lambda_1 \ldots \lambda_n < 0$

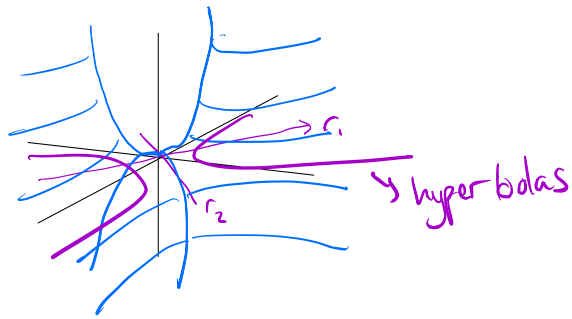
Q pos semi def or PSD

$x^T Q x \geq 0 \quad \forall x \quad \lambda_1 \ldots \lambda_n \geq 0$

also neg semidef NSD

similar... flipped

$x^T Q x \leq 0 \ \forall x \rightarrow \lambda_1 \ldots \lambda_n \leq 0$



worth playing with.

SADDLE FUNCTION

$\lambda_1, \dots, \lambda_n \geq$ or ≤ 0

- SKEW SYMMETRIC MATRIX: $K \in \mathbb{R}^{n \times n}$ $\quad K = -K^T$
  - pure imaginary eigenvalues $\rightarrow$ the eigen values always come in conjugate pairs
    $\lambda_1 = b_1 i,\ \lambda_2 = -b_1 i, \dots$
    $\Rightarrow$ if n is odd at least one $\lambda_i = 0$
  - $K = R \begin{bmatrix} 0 & -b_1 & & & \\ b_1 & 0 & & & \\ & & 0 & -b_2 & \\ & & b_2 & 0 & \\ & & & & \ddots \end{bmatrix} R^T$

    not a diagonalization but related

    related to complex eigenvalues from last week
  - $x^T K x = 0$ for $\forall\, x \in \mathbb{R}^n \rightarrow$ compute directly

SIDE NOTE: $\underset{\substack{\text{general matrix} \\ \in \mathbb{R}^{n \times n}}}{A} = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{skew symmetric}}$

$x^T A x = \underbrace{\tfrac{1}{2} x^T (A + A^T) x}_{\text{sym}} + \underbrace{\tfrac{1}{2} x^T (A - A^T) x}_{0}$

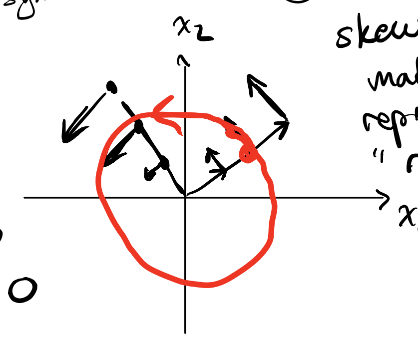
side note:

$f(x) = x^T Q x$

$\nabla f(x) = 2x^T Q$

- $\dot{x} = Kx$

  2 things
  - proportional to $|x|$
  - $\dot{x}^T x = 0$
  - $x^T K^T x = 0$



skew sym matrices represent "rotational vector fields"

- e^{Kt} always a rotation matrix

$\dot{x} = \underset{\sim}{K} x \rightarrow$ integrate $\quad e^{Kt} = R \begin{bmatrix} \cos b_1 t & -\sin b_1 t & & 0 \\ \sin b_1 t & \cos b_1 t & & \\ & 0 & & \ddots \end{bmatrix} R^T$

$R \begin{bmatrix} 0 & -b_1 & \\ b_1 & 0 & \\ & & \ddots \end{bmatrix} R^T$

Lie groups / Lie algebras

if $\lambda = 0$ an eigenvalue of K

then $e^{\lambda t} = 1 \rightarrow$ this eigenvector is called an axis of rotation

have to exist in odd dimensions

MATRIX INVERSES:

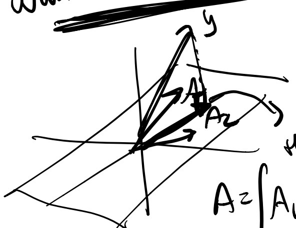
$A \in \mathbb{R}^{n \times m}$ assume smaller matrix dim has full rank

A tall	A square	A fat
$[\ \ ]$	$[\ \ ]$	$[\ \ \ \ \ ]$
cols lin ind	cols & rows ind	rows lin ind

$y = Ax$

A tall: probably no solution because likely $y \notin R(A)$

want to do our best.



assume that y is projected onto the $R(A)$

$A = [A_1 A_2]$

$y \neq Ax$

A square: unique soln

$x = A^{-1} y$

A fat: continuum of solutions

A has a nontrivial nullspace $N(A)$

so if $y = A\bar{x} = A(\bar{x} + z)$, $z \in N(A)$

now want to x with the minimum $|x|$

so find "smallest" norm x

try to minimize ...

$$|y - Ax|^2$$

set $\nabla \uparrow = 0$.

$$\frac{d}{dx}(y^T - x^T A^T)(y - Ax)$$

$$\frac{d}{dx}(y^T y - 2y^T A x + x^T A^T A x) = 0$$

$$-2y^T A + 2x^T A^T A = 0$$

$$\Rightarrow \boxed{x = (A^T A)^{-1} A^T y}$$

Least squares solution

getting x as close as possible

$$[x] = [A^T A]^{-1} [\ A^T\ ] \begin{bmatrix} y \end{bmatrix}$$

estimation

y: measurement signal

x: estimate relatively bad state

spoiler:

"A": observability matrix

$$\begin{bmatrix} y \end{bmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0$$

$$y = A(\bar{x} + z) \quad z \in N(A)$$

z: useless

minimize

$$|\bar{x} + z|^2 = \bar{x}^T \bar{x} + \underbrace{2z^T \bar{x} + z^T z}_{z = 0}$$

$$\bar{x} \in R(A^T)$$

before domain $N(A) \oplus_\perp R(A^T)$

$N(A)$: nothing A

$R(A^T)$: $\bar{x} \in R(A^T)$

$$\bar{x} = A^T w$$

$$y = \underbrace{AA^T}_{\text{invertible}} w \Rightarrow w = (AA^T)^{-1} y$$

$$\boxed{\bar{x} = A^T (AA^T)^{-1} y}$$

minimum norm soln for x

$$(AA^T)^{-1}$$

$$\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} A^T \end{bmatrix} [\ ]\ [y]$$

picking control signal

y: desired state

x: control signal

multiple options for picking

spoiler:

"A": controllability matrix

$$x[t] - A^t x[0] = [A^{n-1}B \cdots AB\ B] \begin{bmatrix} u[0] \\ \vdots \\ u[n-1] \end{bmatrix}$$

LTI Systems: (Linear Time Invariant) $x \in \mathbb{R}^n$

Solutions

Continuous Time: $\dot{x} = Ax + Bu$, $y = Cx$ $\longrightarrow$ $x(t) = \underbrace{e^{At}x(0)}_{\text{drift}} + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$

Discrete Time: $x[t+1] = Ax[t] + Bu[t]$, $y[t] = Cx[t]$ $\quad x[t] = \underbrace{A^t x[0]}_{\text{drift}} + \sum_{\tau=0}^{t-1} A^{t-\tau-1} B u[\tau]$

Solutions

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$x(t) = e^{At}x(0) + \underbrace{\boxed{\left[\; e^{A(t-\tau)}B \;\right]}}_{\text{continuum } [0,t]}^{\bigstar} \begin{bmatrix} u(0) \\ \vdots \\ \tau \\ \vdots \\ u(t) \end{bmatrix}$$

dim x

τ dim

" matrix multiplication along this dim is $\int_0^t$ "

$$x[t] = A^t x[0] + \sum_{\tau=0}^{t-1} A^{t-\tau-1} B u[\tau]$$

$$x[t] = A^t x[0] + \underbrace{\boxed{\left[A^{t-1}B \;\cdots\; AB \;\; B \right]}}_{\text{cols}}^{\bigstar} \begin{bmatrix} u[0] \\ \vdots \\ u[t-1] \end{bmatrix}$$

Controllability / Reachability

question about the range space of $\bigstar$

$$x(t) - e^{At}x(0) \in \mathcal{R}\left(\left[\; e^{A(t-\tau)}B \;\right]\right) \qquad x[t] - A^t x[0] \in \mathcal{R}\left(\left[A^{t-1}B \cdots AB \; B \right]\right)$$

sloppy notation

what is the range of

sloppier...

$$\mathcal{R}\left(\left[\; e^{A(t-\tau)}B \;\right]\right) = \mathcal{R}\left(\left[A^{n-1}B \cdots AB \right]\right)$$

can be shown rigorously.

reason... $e^{A(t-\tau)}$ is a polynomial in A

$$\mathcal{R}\left(\left[\underbrace{A^{t-1}B \cdots}_{\text{redundant}} \;\middle|\; A^{n-1}B \cdots AB \; B \right]\right) \subseteq \mathcal{R}\left(\left[A^{n-1}B \cdots AB \; B \right]\right)$$

check.

Cayley-Hamilton: implies

that $A^t = \beta_{n-1}(t)A^{n-1} + \cdots + \beta_1(t)A + \beta_0(t)I$

for any $t > n-1$

$$\boxed{\left[A^{n-1}B \cdots AB \; B \right]}$$: Controllability matrix

if this spans $\mathbb{R}^n \rightarrow$ controllable or reachable

A system is controllable iff $[A^{n-1}B \cdots AB\ B]$ spans $\mathbb{R}^n$
in both continuous & discrete time

$\underbrace{[A^{n-1}B \cdots AB\ B]}_{\text{full row rank}}$

Controllability Grammians

if M is full row rank $\iff$ (MM^T) is invertible.

$$[\quad M \quad] \begin{bmatrix} \\ M^T \\ \\ \end{bmatrix} = [\quad] \leftrightarrow \text{invertible}$$

CONTINUOUS TIME GRAMMIAN

$$\boxed{W_c = \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} \, d\tau}$$

rigorous

related to the unrigorous thing above

$$W_c = [\ e^{A(t-\tau)}B\ ]_{[0 \quad \tau \quad t]} \begin{bmatrix} B^T e^{A^T(t-\tau)} \end{bmatrix}$$

not rigorous

summing the inner dim is integrating...

DISCRETE TIME GRAMMIAN:

$$W_c = [A^{n-1}B \cdots AB\ B] \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix}$$

same rank

NOTE: could also use

$$\boxed{W_c = [A^{t-1}B \cdots AB\ B] \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{t-1})^T \end{bmatrix}}$$

rank would be the same.

BREAK:

DYNAMICS & COORDINATE TRANSFORMS

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

want to apply new coords

$$x = Tz$$

$$\Rightarrow \quad T\dot{z} = ATz + Bu$$
$$y = CTz$$

similar for discrete time...

what happens to $[A^{n-1}B \cdots AB\ B]$?

$$\dot{z} = \underbrace{T^{-1}AT}_{\bar{A}} z + \underbrace{T^{-1}B}_{\bar{B}} u$$

$$y = \underbrace{CT}_{\bar{C}} z$$

what is

$$[\bar{A}^{n-1}\bar{B} \cdots \bar{A}\bar{B}\ \bar{B}] = [T^{-1}A^{n-1}TT^{-1}B \cdots T^{-1}ATT^{-1}B\ \ T^{-1}B]$$

$$= T^{-1}\underbrace{[A^{n-1}B \cdots AB\ B]}_{\text{doesn't change the row rank since } T \text{ is invertible}}$$

How does controllability break?

1. the vector B written in the eigen coords doesn't contain a specific eigenvector or mode — B is orthogonal to left vectors of A

2. you have repeat eigenvalues and not enough inputs

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1. transform into eigen coords $x = Tz$ — T: cols are eigen vectors

$$\bar{B} = T^{-1}B \Rightarrow \bar{B} = \begin{bmatrix} * \\ * \\ 0 \\ * \\ * \end{bmatrix}$$

$$\Rightarrow T\bar{B} = B$$

$$\underbrace{[v_1 \cdots v_n]}_{T}\begin{bmatrix} * \\ * \\ 0 \\ * \\ * \end{bmatrix} = v_1\bar{B}_1 + \cdots v_{i-1}\bar{B}_{i-1} + v_{i+1}\bar{B}_{i+1} + \cdots$$

$$T^{-1}AT = D \qquad [\bar{A}^{n-1}\bar{B} \cdots \bar{A}\bar{B}\ \bar{B}] = \left[D^{n-1}\begin{bmatrix} * \\ * \\ 0 \\ * \end{bmatrix} \cdots D\begin{bmatrix} * \\ * \\ 0 \\ * \end{bmatrix} \begin{bmatrix} * \\ * \\ 0 \\ * \end{bmatrix}\right]$$

eigen mode can't be controlled $\rightarrow = [0\ 0\ 0\ 0\ 0]$

$A = \underset{\text{directions of evolution of the sys.}}{T}\, D\, \underset{\text{input directions}}{T^{-1}}$ $\qquad \begin{bmatrix} * \\ * \\ 0 \\ * \end{bmatrix} = T^{-1}B$

② in the eigenvector coordinates

$$\left[D^{n-1}\bar{B} \cdots D\bar{B}\ \bar{B}\right] = \left[\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_1\end{bmatrix}^{n-1}\begin{bmatrix}\bar{b}_1\\ \bar{b}_2\end{bmatrix} \atop (D')^{n-1}\bar{B}'\right] \cdots \begin{bmatrix}\bar{b}_1\\ \bar{b}_2\\ \bar{B}'\end{bmatrix}$$

ratios of the same always

$$\bar{B} = \begin{bmatrix}\bar{b}_1\\ \bar{b}_2\\ \bar{B}'\end{bmatrix} \quad D = \begin{bmatrix}\lambda_1\ 0 & 0\\ 0\ \lambda_1 & \\ 0 & D'\end{bmatrix}$$

PBH TEST:

$$\left[\lambda I - A \,\middle|\, B\right] \Longleftrightarrow \text{the system is controllable}$$

if matrix full row rank (spans $\mathbb{R}^n$)
for $\underbrace{\lambda = \lambda_1 \cdots \lambda_n}_{\text{eigenvalues}}$

MINIMUM NORM CONTROL INPUTS:

$$\underbrace{x(t) - e^{At}x(0)}_{\substack{\text{target}\\ \text{"y"}}} = \underbrace{\left[\ e^{A(t-\tau)}B\ \right]}_{\substack{\text{where we can get}\\ \text{"A"}}} \underbrace{\begin{bmatrix}u(0)\\ \vdots\\ \tau\\ \vdots\\ u(t)\end{bmatrix}}_{\substack{\text{input}\\ \text{"x"}}}$$

$$\underbrace{x[t] - A^t x[0]}_{\substack{\text{"y"}\\ \text{target}}} = \underbrace{\left[A^t B' \cdots AB\ B\right]}_{\substack{\text{"A"}\\ \text{where we can get to}}} \underbrace{\begin{bmatrix}u[0]\\ \vdots\\ u[t-1]\end{bmatrix}}_{\substack{\text{"x"}\\ \text{input}}}$$

FROM BEFORE: SOLVING $y = Ax$
minimum norm solution was $x = \boxed{A^T(AA^T)^{-1}} y$

$$y = Ax = [\ A\ ]\left[A^T\right](AA^T)^{-1}y = y$$

[ ]

CONT. TIME MIN NORM CONT.

$$\boxed{u(t) = B^T e^{A^T t} W_c^{-1}\left[x(t) - e^{At}x(0)\right]}$$

will double check. $n[e^{At}]\ n[W_c]^{-1}$ n-dim vector
$B \in \mathbb{R}^{n \times m}$ $B^T \in \mathbb{R}^{m \times n}$

DISCRETE TIME MIN NORM CONT.

$$\boxed{\begin{bmatrix}u[0]\\ \vdots\\ u[t-1]\end{bmatrix} = \begin{bmatrix}B^T\\ B^T A^T\\ \vdots\\ B^T (A^T)^{t-1}\end{bmatrix}\left[W_c\right]^{-1}\left[x[t] - A^t x[0]\right]}$$

OBSERVABILITY:

HOW DO WE FIGURE OUT WHAT STATE WE STARTED IN $x(0)$ given A SET OF OUTPUTS $y(t)$ $[0, t]$

Measurement Eqn:

$$y = Cx$$

$$= [\leftarrow C \rightarrow] \begin{bmatrix} | \\ x \\ | \end{bmatrix}$$

DISCRETE TIME CASE:

$$x(t) = e^{At} x(0) + \begin{bmatrix} & e^{A(t-\tau)} B & \end{bmatrix} \begin{bmatrix} u(0) \\ \uparrow \\ \tau \\ \downarrow \\ u(t) \end{bmatrix} \qquad x[t] = A^t x[0] + \begin{bmatrix} A^{t-1}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ \vdots \\ u[t-1] \end{bmatrix}$$

$$y(t) = Cx(t) = Ce^{At} x(0) + C\underbrace{[\text{---}]}_{\text{know}} \underbrace{\begin{bmatrix} u(0) \\ \vdots \\ u(t) \end{bmatrix}}_{\text{know}} \qquad y[t] = CA^t x[0] + C\underbrace{[\text{---}]}_{\downarrow} \underbrace{\begin{bmatrix} u[0] \\ | \\ u[t-1] \end{bmatrix}}_{\text{know}}$$

$$\underbrace{y(t) - C[\text{---}] \begin{bmatrix} u(0) \\ \vdots \\ u(t) \end{bmatrix}}_{\bar{y}(t)} = Ce^{At} x(0) \qquad \underbrace{y[t] - C[\text{---}] \begin{bmatrix} u[0] \\ \\ u[t] \end{bmatrix}}_{\bar{y}[t]} = CA^t x[0]$$

$$\bar{y}(t) = Ce^{At} \underbrace{x(0)}_{\text{estimate for initial state}} \qquad \bar{y}[t] = CA^t \underline{x[0]}$$

$$\downarrow$$

$$\begin{bmatrix} \bar{y}[0] \\ \bar{y}[1] \\ \vdots \\ \bar{y}[t] \end{bmatrix} = \begin{bmatrix} C\,x[0] \\ CA\,x[0] \\ \vdots \\ CA^t\,x[0] \end{bmatrix}$$

also check

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

full column rank.

similarly to control. for continuous time ←

+

$$\begin{bmatrix} \bar{y}[0] \\ \bar{y}[1] \\ \vdots \\ \bar{y}[t] \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t} \end{bmatrix} \underline{x[0]}$$

require that ↑ has full column rank.

observability matrix

LEAST SQUARES ESTIMATE

$$\underbrace{\begin{bmatrix} \bar{y}[0] \\ \vdots \\ \bar{y}[t] \end{bmatrix}}_{\text{might not exactly lie in the range of } *} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t} \end{bmatrix}^{*}}_{\text{tall}} \underline{x[0]}$$

again Cayley Hamilton only need to check that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is full column rank

can estimate any initial condition

the system is **observable**

least squares estimate of $x[0]$ is

$$x[0] = \left(\left[C^T A C^T \cdots (A^t)^T C^T \right] \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t} \end{bmatrix} \right)^{-1} \left[C^T A C^T \cdots (A^t)^T C^T \right] \begin{bmatrix} \bar{y}[0] \\ \vdots \\ \bar{y}[t] \end{bmatrix}$$

PBH TEST: $\begin{bmatrix} C \\ \lambda I - A \end{bmatrix}$ has full column rank for $\underbrace{\lambda = \lambda_1 \dots \lambda_n}_{\text{eigenvalues}}$ $\Leftrightarrow$ observable

PREVIEW: NEXT WEEK <u>LINEAR STATE FEEDBACK</u>

choose control as linear function of the state

$u(t) = K x(t)$

$\dot{x} = Ax(t) + Bu(t) = (A+BK)\,x(t)$

choose $A+BK$ to have properties that we want like specific eigenvalues $\Rightarrow$

STATE ESTIMATION PROBLEM:

given $y(t) \rightarrow$ estimate $x(t)$ as we go...

MODEL SYSTEM: $\dot{\hat{x}} = A\hat{x} + Bu + L(\overset{\text{expected meas}}{C\hat{x}} - \overset{\text{meas}}{y})$

$\hat{x}$: model of the state $\qquad \dot{\hat{x}} = (\underbrace{A+LC}) \hat{x} + Bu - Ly$

want $\hat{x} \rightarrow x$

---

x, $e = x - \hat{x}$ error between model & actual state

for control use $u = K\hat{x}$ $\leftarrow$ because we don't have access to x

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} - A\hat{x} - \cancel{Bu} - L(C\hat{x} - y)$$

$$= A(x - \hat{x}) - LC(x-e) + Ly$$

$$= Ae - LC(x-e) + Ly$$

$$= (A+LC)e - LCx + Ly \qquad y = Cx$$

$$\dot{e} = (A+LC)e$$

$$\dot{x} = Ax + Bu = Ax + BK\hat{x} = Ax + BK(x-e)$$

$$= (A+BK)x - BKe$$

$$\dot{e} = (A+LC)e$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix}}_{\text{eigen values of a block diagonal matrix are just the eigenvalues of each block together}} \begin{bmatrix} x \\ e \end{bmatrix} \quad \begin{matrix} \leftarrow \text{actual sys dyn.} \\ \leftarrow \text{error dyn.} \end{matrix}}$$

summary:

can design K to stabilize/control the system

can design L to stabilize the error dynamics

add $L(C\hat{x}-y)$ as an input to the error dynamics

and use $u = K\hat{X}$ $\longrightarrow$ then the error goes to 0 & we stabilize the system separation principle of estimation & control ← works in linear systems.