Homework 3 Questions:

Question 1d & e : $f: \mathbb{R}^n \to \mathbb{R}$ $\quad \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$

e) $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right)\right.$$ general.

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} & & \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_n} \\ \vdots & & \\ \frac{\partial}{\partial x_n}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n}\frac{\partial f}{\partial x_n} \end{bmatrix}$$

d) A is not sym.

not true that $\frac{\partial^2 f}{\partial x_i \partial x_j} \neq A_{ij}$

is equal to $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{2}(A_{ij} + A_{ji})$

Homework 2 questions:

uniqueness of eigenvectors...

$$A = PDP^{-1} = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^{-1}$$

$E \in \mathbb{C}^{n \times n}$ (diagonal) $\quad E^{-1} \in \mathbb{C}^{n \times n}$ (diagonal)

$$E = \begin{bmatrix} r_1 e^{i\phi} & & \\ & \ddots & \\ & & \ddots \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} \frac{1}{r_1} e^{-i\phi} & \\ & \ddots \end{bmatrix}$$

$$PE = \begin{bmatrix} r_1 v_1 e^{i\phi} \cdots \\ \\ \end{bmatrix} \qquad E^{-1}P^{-1} = (PE)^{-1}$$

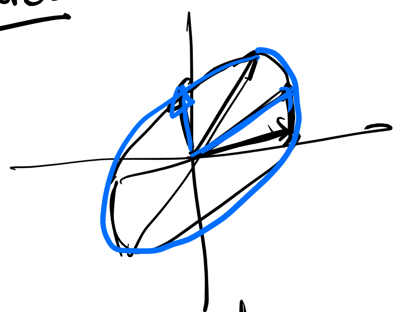
$r_1 v_1 e^{i\phi} \ldots$

$$E = \begin{bmatrix} e^{i\phi} & \\ & e^{-i\phi} \end{bmatrix} \text{ or } \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix}$$

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix} = \begin{bmatrix} u' & v' \end{bmatrix} \qquad E^{-1} = \begin{bmatrix} c(-\phi) & -s(-\phi) \\ s(-\phi) & c(-\phi) \end{bmatrix}$$

$u' = c\phi u + s\phi v$ etc.

Pictures:



Condition Number:

$y = Ax$ A ill conditioned — essentially cols together; for a particular y, x is huge

$x = A^{-1} y$

LECTURE NOTES:

Clarifications:

1) $\dot{x} = Ax + Bu \to \lambda = a + bi$ eval stability $\underline{Re(\lambda)} < 0$

$x[t+1] = \bar{A}x[t] + \bar{B}u[t]$ $\bar{\lambda}$ eval $\bar{A}$ stability $|\bar{\lambda}| < 1$

look at $\bar{A} = e^{A\Delta t}$ consistent.

$(\bar{A})^t$ if $|\lambda_i| < 1 \ \forall i$ $\bar{\lambda}_i^t \to 0$ if $|\bar{\lambda}_i| < 1$

2) if $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ evals$(M) =$ evals$(A) \cup$ evals(D)

evecs: $\lambda v = Av$ $\quad M\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v \\ 0 \end{bmatrix}$

$\mu w = Dw$ $\quad M\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix} = \mu \begin{bmatrix} x \\ w \end{bmatrix}$

$\Rightarrow \mu x = Ax + Bw \Rightarrow x = (\mu I - A)^{-1} Bw$

subtlties if μ is also eval of A

general picture.

Interpretations of Controllability

Not controllable if

- B is $\perp$ to a left evec.

left eigenvectors = "input directions"

$A = PDP^{-1}$

$$\begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} = P\begin{bmatrix} D^{n-1}P^{-1}B & \cdots & DP^{-1}B & P^{-1}B \end{bmatrix}$$

condition $\Rightarrow$ $P^{-1}B = \begin{bmatrix} * \\ 0 \\ * \\ \vdots \end{bmatrix} \rightarrow D^k P^{-1}B = \begin{bmatrix} * \\ 0 \\ \vdots \end{bmatrix}$

$$= P \begin{array}{l} \\ \text{ith entry} \rightarrow \end{array} \begin{bmatrix} * & \cdots & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} n \Rightarrow$$

one the columns never shows up

$$e_i^T P^{-1} P \begin{bmatrix} * & \cdots & & \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

$\Rightarrow$ controllability matrix has a left nullspace

$\Rightarrow$ not full row rank.

Transform Coords on dynamics

$\dot{x} = Ax + Bu$ plug in $x = Tz \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$

in z coords $\left[(T^{-1}AT)^{n-1} T^{-1}B \ \cdots \ \right]$

$$\left[T^{-1}A^{n-1}B \ \cdots \ T^{-1}AB \ T^{-1}B \right]$$

$$T^{-1}\left[A^{n-1}B \ \cdots \ AB \ B \right]$$

Not observable...

$C \perp$ to a right eigenvector

"output direction"

Not controllable if
not enough inputs for repeats eigenvals.

example $\lambda_1 = \lambda_2$ $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \ddots \end{bmatrix}$ $\quad \bar{B} = \mathbb{R}^{n\times 1}$

in the eigen vector coords...

$$\left[D^{n-1}\bar{B} \ \cdots \ D\bar{B} \ \bar{B} \right] \qquad \bar{B} = \begin{bmatrix} \bar{b} \\ * \\ * \\ * \end{bmatrix} \quad \bar{b} \in \mathbb{R}^2$$

if you choose $w \in \mathbb{R}^2$
s.t. $w^T \bar{b} = 0$

left multiply...

$$\begin{bmatrix} w^T & 0 \end{bmatrix} \begin{bmatrix} \lambda_1^{n-1} I \bar{b} & \lambda_1^{n-2} I \bar{b} & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} \lambda_1^{n-1} w^T \bar{b} & \lambda_1^{n-2} w^T \bar{b} & \cdots \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

w^T can't be $\perp$ to $\begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} \bar{b}$ & $\begin{bmatrix} \lambda_1^{n-2} & \\ & \lambda_2^{n-2} \end{bmatrix} \bar{b}$ etc.

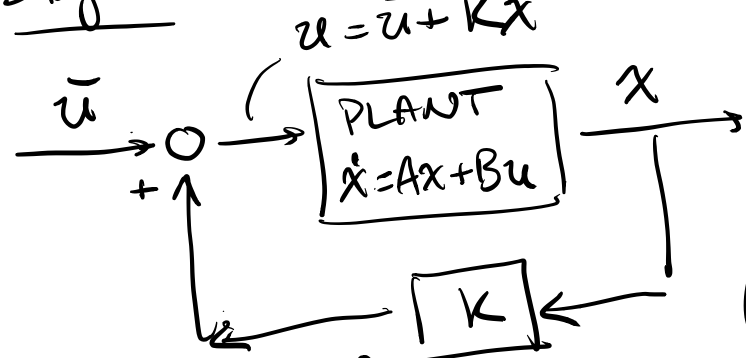
if $\lambda_1 \neq \lambda_2$

but if $\lambda_1 = \lambda_2$ then

FEEDBACK CONTROL & OBSERVER DESIGN:

$\dot{x} = Ax + Bu$ want $u = \bar{u} + Kx \Rightarrow \dot{x} = \underbrace{(A+BK)}_{\text{stable}}x + B\bar{u}$

Block Diagram:



want to shape evals of $(A+BK)$ (& eigenvectors if we can)

What if no access to x?

measurement $y = Cx$ where C is not invertible

1. Model state: $\hat{x}$
2. Use y as input to the model dynamics to drive $\hat{x}$ to x
3. Use control $\underline{u = \bar{u} + K\hat{x}}$

} observer design

ACTUAL STATE

$$\dot{x} = Ax + Bu$$
$$= Ax + BK\hat{x} + B\bar{u}$$
$$y = Cx$$

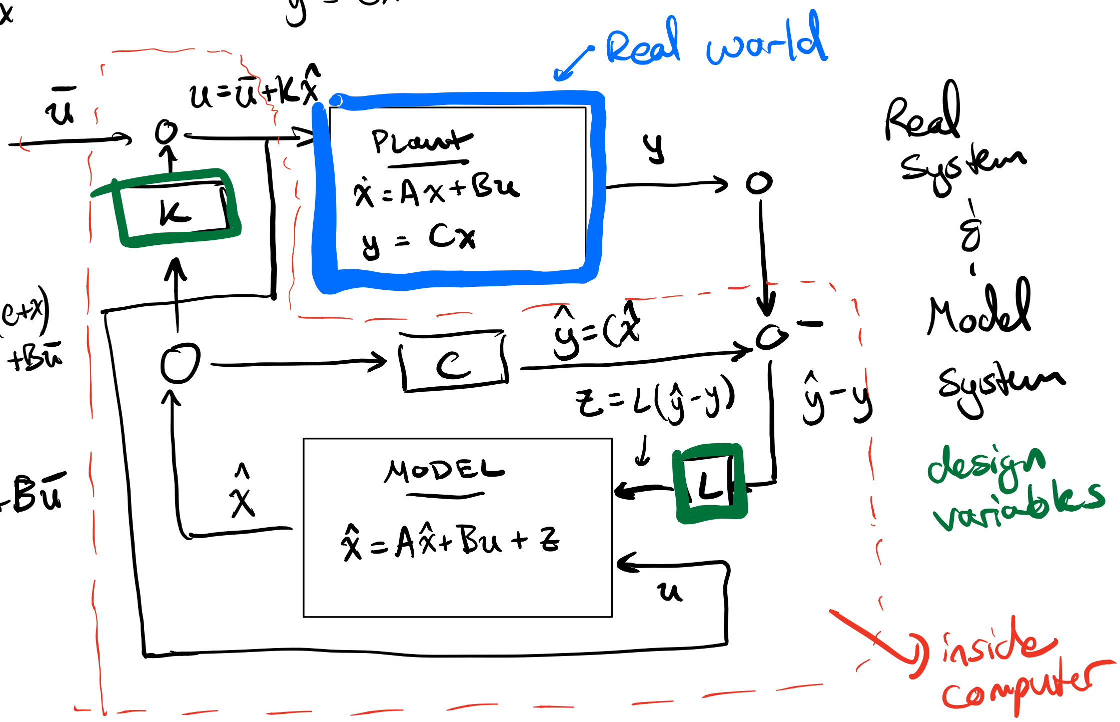
MODEL STATE

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\overset{\text{exp meas}}{\hat{y}} - \overset{\text{actual meas}}{y})$$
$$= A\hat{x} + Bu + L(C\hat{x} - Cx)$$
$$\hat{y} = C\hat{x}$$

ERROR DYNAMICS $e = \hat{x} - x$

$$\dot{e} = \dot{\hat{x}} - \dot{x} = A(\hat{x} - x) + L(C\hat{x} - Cx)$$
$$\dot{e} = (A + LC)e.$$

$$\hat{x} = e + x$$
$$\dot{x} = Ax + BK(e+x) + B\bar{u}$$
$$\dot{x} = (A+BK)x + BKe + B\bar{u}$$



Full Dynamics: (in terms of x & e)

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A+BK & BK \\ 0 & A+LC \end{bmatrix}}_{} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \bar{u}$$

evals of this full system matrix just depend on evals of $A+BK$ & evals of $A+LC$

$\Rightarrow$ design K to stabilize $A+BK$ & design L to stabilize $A+LC$ separately

$\Rightarrow$ separation principle

mathematically the same $A+BK \leftrightarrow A^T + C^T L^T$

How to pick K & L:

simplest case where $B \in \mathbb{R}^{n \times 1}$ ($C \in \mathbb{R}^{1 \times n}$)

$\Rightarrow$ choose the evals of the closed loop system.

picking evals $\rightarrow$ choosing closed loop characteristic polynomial

want $\det(\lambda I - A - BK) \overset{②}{=} \prod_i (\lambda - \lambda_i) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1 \lambda + \beta_0$

design choice (λ_i) — can compute

Assume ① $\det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$

pick K to change char poly from ① to ② given

Consider system of form $\dot z = \bar A z + \bar B u$

where
$$\bar A = \begin{bmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & 0 \end{bmatrix} \quad \bar B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

controllable canonical form

$\det(\lambda I - \bar A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0$

if we choose $\bar K = [\alpha_{n-1} - \beta_{n-1} \;\; \cdots \;\; \alpha_0 - \beta_0]$

/

use this gain → closed loop char poly is as we want

Question:
can we find a similarity transform that converts our original system to this form

$\dot{x} = Ax + Bu$

want T invertible s.t. $x = Tz$

$T^{-1}AT = \bar{A}$ & $T^{-1}B = \bar{B}$

Answer: yes if the system is controllable

Note: $[\bar{A}^{n-1}\bar{B} \cdots \bar{A}\bar{B}\ \bar{B}]$ is always invertible

if T exists... then we have

$$\underbrace{[\bar{A}^{n-1}\bar{B} \cdots \bar{A}\bar{B}\ \bar{B}]}_{\bar{M}} = T^{-1}\underbrace{[A^{n-1}B \cdots AB\ B]}_{M}$$

if M is invertible ⟺ the system is controllable

then ... $\boxed{T^{-1} = \bar{M}M^{-1}}$ ← similarity transform

Feedback: $Kx = KTz = \bar{K}z$

$$\Rightarrow \boxed{K = \bar{K}T^{-1} = \bar{K}\bar{M}M^{-1}}$$

- pole placement using controllable canonical form
- another version of this is Ackermann's Formula

what if $B \in \mathbb{R}^{n \times m}$?

- multiple choices for K that give the desired eigenvals → freedom to choose eigenvectors
- place command works multi input systems.

pick eigenvalues.

→ chooses eigenvectors to minimize the condition # of X where the cols of X are the eigenvectors of $A+BK$

$$X^T X = \begin{bmatrix} - x_1^T - \\ - x_n^T - \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} x_1^T x_1 & \cdots & x_1^T x_n \\ x_n^T x_1 & \cdots & x_n^T x_n \end{bmatrix} \Big\} \to \text{defines the shape of } X$$

Condition

Singular Value Decomposition

For any $X \in \mathbb{C}^{n \times m}$

$$X = \underset{\in \mathbb{C}^{n \times n}}{U} \Sigma \underbrace{V^*}_{\in \mathbb{C}^{m \times m}}$$

$$\begin{array}{cccc} & & m & m \\ n & \underbrace{\begin{bmatrix} U \end{bmatrix}}_{\text{another rotation}} n & \underbrace{\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}}_{\text{stretching}} m & \underbrace{\begin{bmatrix} V^* \end{bmatrix}}_{\text{first rotation}} \end{array}$$

Σ: diagonal & positive

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

σ_i: non zero evals of $(X^T X)^{1/2}$ & $(X X^T)^{1/2}$

or equivalently square roots of the nonzero evals of $X^T X$ & $X X^T$

Polar Decomposition: if X is square invertible...

$$X = \underbrace{X (X^T X)^{-1/2}}_{\text{ROT \& REFLECTION}} \underbrace{(X^T X)^{1/2}}_{\text{pos def}} = \underbrace{(X X^T)^{1/2}}_{\text{pos def}} \underbrace{(X X^T)^{-1/2} X}_{\text{rot \& reflection}}$$

ROT & REFLECTION: cols are orthonormal

like a rotation but might not have determinant of 1

pos def: determined by relative shape of cols

pos def: by relative shape of rows

rot & reflection

like complex #s $z = re^{i\phi}$

$$X = \underset{\text{rot. reflection}}{R}\, \underset{\to \text{pos def}}{P}$$

SVD: works on matrix of any dimension

$$X = \left[U_1 \middle| U_2 \right] \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array} \right] \left[\begin{array}{c} V_1^* \\ \hline V_2^* \end{array} \right] \Rightarrow$$

U_1: orthonormal basis for $R(X)$

U_2: orthonormal basis for $N(X^*)$

V_1: orthonormal basis for $R(X^*)$

V_2: orthonormal basis for $N(X)$

Condition # $\dfrac{\sigma_{max}(X)}{\sigma_{min}(X)}$

if X is invertible:

$$X^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^*$$

$$\Sigma^{-1} = [\Sigma] = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_k} \end{bmatrix}$$

if $\sigma_{max} >> \sigma_{min}$ $\rightarrow \begin{bmatrix} \text{huge} & \alpha \\ \alpha & \text{small} \end{bmatrix}$

closed loop matrix

$$A + BK = XDX^{-1}$$

if X is poorly conditioned = relatively close to 0 singular value then X^{-1} gets huge in certain directions

$$(A+BK)x(0) = XD\underline{X^{-1}x(0)}$$

pick D to have evals you want

could be huge depending on $x(0)$

the amplitude of $X^{-1}x(0)$ could be big if X is poorly conditioned.

$\rightarrow$ place command for multi input systems chooses X to be as well conditioned as possible.

"Robust Pole Assignment in Linear State Feedback (1985) Kautsky, Nichols Van Dooren.

LINEAR TIME VARYING SYSTEMS & LINEARIZATION

CONTINUOUS TIME

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

State transition Matrix: $e^{A(t-t_0)} \rightarrow \phi(t,t_0)$

solution to $\dot{\phi}(t,t_0) = A(t)\phi(t,t_0)$

with i.c. $\phi(t_0,t_0) = I$

- $\phi(t,t_0) = I$
- $\phi(t_0,t) = \phi^{-1}(t,t_0)$
- $\phi(t,t_0) = \phi(t,t')\phi(t',t_0)$
- $\dot{\phi}(t,t_0) = A(t)\phi(t,t_0)$

$\phi(t,t_0)$: map between initial conds and conds at time t

$x(t) = \phi(t,0)x(0)$ if no control input

with control input.

$$x(t) = \phi(t,0)x(0) + \int_0^t \phi(t,\tau)B(\tau)u(\tau)d\tau$$

General form for computing this is Peano-Baker series → nested integrals ugly.

DISCRETE TIME

$$x[t+1] = A[t]x[t] + B[t]u[t]$$

state transition matrix: $A^t \rightarrow A[t] \times \cdots \times A[0]$

$$x[t+1] = A[t]\cdots A[0]x[0] + \sum_{\tau=0}^{t-1} A[t]\times\cdots\times A[\tau+1]B[\tau]u[\tau] + B[t]u[t]$$

LINEARIZATION OF NONLIN. DYN

$$\dot{x} = f(x,u,t)$$

nominal control trajectory: $\bar{u}(t)$

plug into dynamics to get nominal state trajectory: $\bar{x}(t)$

solution to integrating the dynamics

will apply control $u = \bar{u} + \Delta u = \bar{u} + K\Delta x$

addition to control to stabilize around trajectory

$$x(t) = \bar{x}(t) + \underbrace{\Delta x(t)}_{\text{perturbations to the state}}$$

$$\dot{x} = \dot{\bar{x}} + \dot{\Delta x} = f(\bar{x}+\Delta x, \bar{u}+\Delta u, t)$$

$$\cancel{\dot{\bar{x}}} + \Delta\dot{x} = \cancel{f(\bar{x},\bar{u},t)} + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}\bar{u}t} \Delta x + \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u},t} \Delta u$$

$$\Delta\dot{x} = \boxed{\frac{\partial f}{\partial x}}\Bigg|_{\bar{x},\bar{u},t} \Delta x + \left.\frac{\partial f}{\partial u}\right| \Delta u \Rightarrow$$

LIN. TIME VARYING PERTURBATION DYNAMICS

$$\Delta x[t+1] = \Delta x[t] + \Delta t \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u},t} \Delta x[t] + \Delta t \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u},t} \Delta u[t]$$

$$= \left[I + \Delta t \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u},t} \right] \Delta x[t] + \Delta t \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u},t} \Delta u[t]$$

FORWARD EULER INTEGRATION.

IN THE LTI CASE:

A in continuous time $\rightarrow$ $e^{A\Delta t}$ discrete time.

$\left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u},t}$ in cont time $\rightarrow$ $I + \Delta t \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u},t}$ $\Leftarrow$ first terms in the Taylor exp of the state trans matrix.

LYAPUNOV STABILITY THEORY:

IDEA: define an "energy" function for a system & then show that it decreases along trajectories of the system $\rightarrow$ way to show stability.

used in linear systems & nonlinear systems

$\rightarrow$ related to linear quadratic regulator (LQR) the cost-to-go in the LQR problem acts like a Lyapunov function

LEMMA: function $F(t)$ s.t.

$$\dot{F}(t) \leq \lambda F(t) \Rightarrow F(t) \leq e^{\lambda t} F(0)$$

PROOF: define $u(t) = e^{-\lambda t} F(t)$

take deriv. $\frac{d}{dt} u(t) = -\lambda e^{-\lambda t} F(t) + e^{-\lambda t} \dot{F}(t) \leq \underbrace{-\lambda e^{-\lambda t} F(t) + \lambda e^{-\lambda t} F(t)}_{\text{terms cancel}}$

$$\dot{u}(t) \leq 0$$

$$u(t) = e^{-\lambda t} F(t) \leq u(0) = F(0)$$

$$\Rightarrow e^{-\lambda t} F(t) \leq F(0) \Rightarrow F(t) \leq e^{\lambda t} F(0)$$

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if we know the derivative of a scalar function is always bounded by some scalar times the function value → then we can use that scalar value λ as decay rate that bounds the function value

→ useful when $\lambda < 0$

because $e^{\lambda t} \to 0 \Rightarrow F(t) \to 0$

$F(t)$: energy type function

Apply to linear systems:

Consider function $F(x(t)) = x(t)^T P x(t)$ where $P = P^T > 0$ is symmetric PD

and x evolves according to $\dot{x} = Ax$

$$\dot{F} = \frac{\partial F}{\partial x}\dot{x} = x^T P \dot{x} + \dot{x}^T P x$$

"Lie derivative" $= x^T PAx + x^T A^T P x = x^T(A^T P + PA)x$

Note: for a matrix $Q = Q^T > 0$

$$\lambda_{min}(Q)|x|^2 \leq x^T Q x \leq \lambda_{max}(Q)|x|^2$$

suppose $A^T P + PA = -Q$ for some $Q = Q^T > 0$

$$\dot{F}(x(t)) = -x^T Q x \leq -\lambda_{min}(Q)|x|^2 \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^T P x$$

from $\lambda_{min}(Q)|x|^2 \leq x^T Q x$

$$\frac{x^T P x}{\lambda_{max}(P)} \leq |x|^2$$

Note: the negative signs.

$$\dot{F}(x(t)) \leq \underbrace{-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}}_{<0} F(t) \Rightarrow F(t) \leq e^{-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}t} F(0)$$

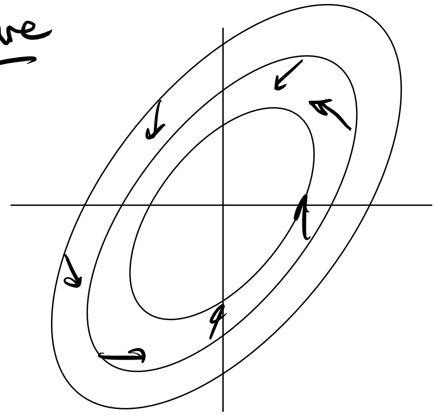
Note: $F(x(t)) = x^T P x > 0$ for $x \neq 0$

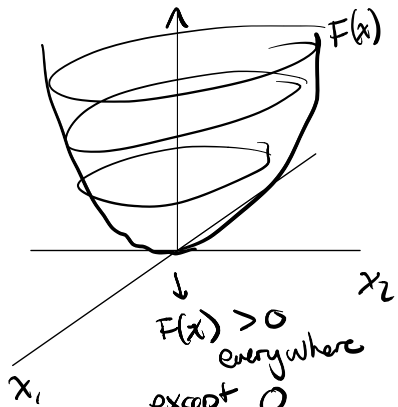
$F(0) = 0$

since $-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} < 0 \Rightarrow e^{-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}t} F(0) \to 0 \to F(x(t)) \to 0$

since $F = 0$ only at $x(t) = 0 \to x(t) \to 0$

Picture





if we show $\dot{x}$ always makes $F(x)$ decrease then we can use this to show stability.