Homework comments:

- systems not realistic
- HW3 interpreting eigenvectors - go easy on grading
- HW4: don't $e \neq 0$
- road disturbance: the 2nd input is road disturbance
  $w \leftarrow$ set to 0
  just use the first column

Vector Calculus:

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x)$

what is a derivative? deriv. maps $\Delta x \to \Delta f$

$$\Delta f = \frac{\partial f}{\partial x}\Delta x = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \sum_i \frac{\partial f}{\partial x_i}\Delta x_i$$

Example:

$f(x) = Cx \to \frac{\partial f}{\partial x} = C \qquad \frac{d}{dx} cx = c$

$f(x) = x^T Q x$

product rule: "perturb ea. variable separately"

$$\Delta f = \underbrace{\Delta x^T Q x + x^T Q \Delta x}_{\text{collect } \Delta x \text{ terms}} = \underbrace{x^T (Q + Q^T)}_{\frac{\partial f}{\partial x}} \Delta x$$

sometimes see $2x^T Q$ only true if $Q = Q^T$

Note: always assume $Q = Q^T$ because if not

$$x^T Q x = x^T \Big(\underbrace{\tfrac{1}{2}(Q+Q^T)}_{\text{sym}}\Big) x + \cancel{x^T \Big(\underbrace{\tfrac{1}{2}(Q-Q^T)}_{\text{skew sy}}\Big) x}^{\,0}$$

what if $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\underset{\in \mathbb{R}^m}{\Delta f} = \frac{\partial f}{\partial x}\Delta x = \begin{array}{c c} & n \\ m & \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \end{array} = \begin{bmatrix} | & & | \\ \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ | & & | \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} - \frac{\partial f_1}{\partial x} - \\ \vdots \\ - \frac{\partial f_m}{\partial x} - \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

Linearization: $\dot{x} = f(x, u, t)$, $x(0) = x_0$

some nominal control: $\bar{u}$

→ " state traj: $\bar{x}$

$$\dot{\bar{x}} + \dot{\Delta x} = f(\bar{x} + \Delta x, \bar{u} + \Delta u, t)$$

1st order approx

$$= f(\bar{x}, \bar{u}, t) + \frac{\partial f}{\partial x}\bigg|_{\bar{x}, \bar{u}} \Delta x + \frac{\partial f}{\partial u}\bigg|_{\bar{x}, \bar{u}} \Delta u$$

$$\dot{\Delta x} = \frac{\partial f}{\partial x}\bigg|_{\bar{x}, \bar{u}} \Delta x + \frac{\partial f}{\partial u}\bigg|_{\bar{x}, \bar{u}} \Delta u$$

$x \in \mathbb{R}^n$ $\quad$ $u \in \mathbb{R}^m$

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & & \\ & & \end{bmatrix} \end{array} \qquad \begin{array}{cc} & m \\ n & \begin{bmatrix} \quad & \quad \\ & \\ & \end{bmatrix} \end{array}$$

Example:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{v} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta\, v \\ \sin\theta\, v \\ \frac{1}{m} u \\ \omega \end{pmatrix} \rightarrow f(\underbrace{z_1, z_2, v, \theta}_{\text{states}}, \underbrace{u, \omega}_{\text{controls}})$$

$$\frac{\partial f}{\partial x} = \begin{array}{cc} & 4 \\ 4 & \begin{bmatrix} 0 & 0 & \cos\theta & -\sin\theta\, v \\ 0 & 0 & \sin\theta & \cos\theta\, v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{pmatrix} \Delta z_1 \\ \Delta z_2 \\ \Delta v \\ \Delta\theta \end{pmatrix} \qquad \frac{\partial f}{\partial u} = \begin{array}{cc} & 2 \\ 4 & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \Delta u \\ \Delta\omega \end{bmatrix}$$

Derivative w.r.t. a matrix?

$f : \mathbb{R}^{m \times n} \to \mathbb{R}$ want map ΔX to Δf

Example: $f(X) = \boxed{Tr(C^T X)} = \sum_{ij} C_{ij} X_{ij}$

$X, C \in \mathbb{R}^{m \times n}$ "dot product of 2 matrices"

$$\boxed{\frac{\partial f}{\partial X_{ij}} = C_{ij}}$$

$$\Delta f = Tr(C^T \Delta X)$$

$$\boxed{\frac{\partial f}{\partial X} = C}$$

true $\Delta f = Tr\left(\frac{\partial f}{\partial X}^T \Delta X\right)$

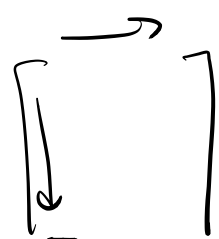
not true $\Delta f \neq \frac{\partial f}{\partial X} \Delta X \to$ dimensions might not even work

Example:

$f = x^T Q x$ $\quad \frac{\partial f}{\partial Q} = ?$ $\quad f = Tr(x^T Q x) = \boxed{Tr(x x^T Q)}$

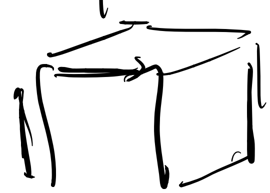
$$\Rightarrow \frac{\partial f}{\partial Q} = x x^T \Rightarrow \Delta f = Tr\left(\frac{\partial f}{\partial x}^T \Delta Q\right)$$

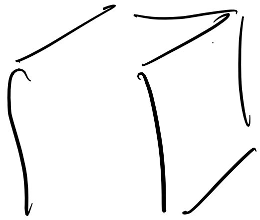
$f(x) = [\,]$ $\longrightarrow$ $\frac{\partial f}{\partial x} = [\longrightarrow]$ $\longrightarrow$ $\frac{\partial^2 f}{\partial x}$



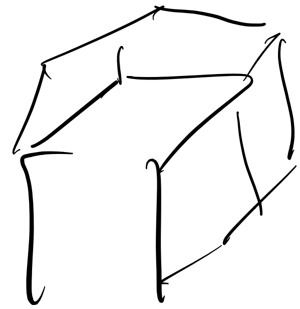
$f(x) = [\,|\,]$ $\longrightarrow$ $\frac{\partial f}{\partial x} = [\;]$

$\frac{\partial^2 f}{\partial x} =$



$f(x) = \begin{bmatrix} \ \end{bmatrix} \rightarrow \frac{\partial f}{\partial x} =$ 

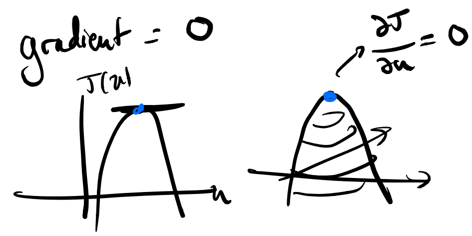
$f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$

$\begin{bmatrix} \ \end{bmatrix} \longrightarrow \frac{\partial f}{\partial x}$  ← 4 tensor

Lagrange Multipliers:

unconstrained opt: $\Rightarrow \frac{\partial J}{\partial u} = 0$ gradient = 0

$\min_u J(u)$



equality const opt:

$\min_u J(u)$

s.t. $g(u) = 0$ ↑ vector

$\mathbb{R}^k \ni \lambda$

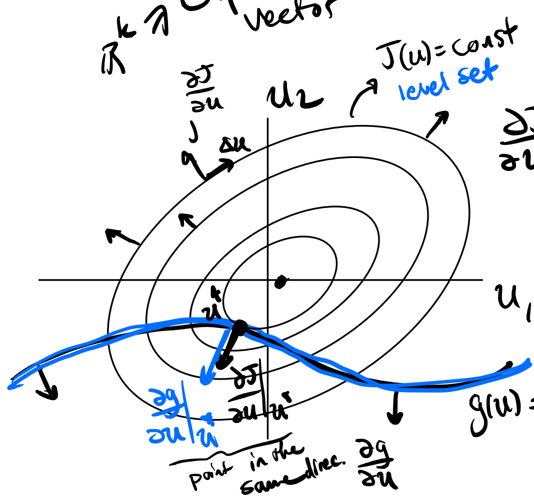
Lagrangian: $\lambda \in \mathbb{R}^k$

$$L(u, \lambda) = \underline{J(u)} + \underline{\lambda^T g(u)}$$

$\frac{\partial L}{\partial u} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$\downarrow$ $\frac{\partial J}{\partial u} + \lambda^T \frac{\partial g}{\partial u}$

$\downarrow$ $g(u) = 0$ constraint satisfaction

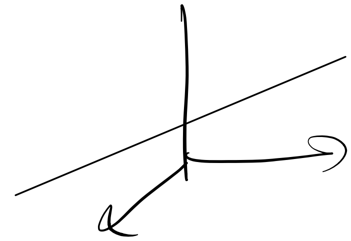


$\frac{\partial J}{\partial u}$: $\perp$ to level sets ← $\Delta J = \underset{\sim}{\frac{\partial J}{\partial u} \Delta u} = 0$

$$\rightarrow \frac{\partial J}{\partial u} + \underset{\substack{\uparrow \\ \text{lagrange multiplier}}}{\lambda^T} \frac{\partial g}{\partial u} = 0 \leftarrow \frac{\partial J}{\partial u} \ \& \ \frac{\partial g}{\partial u} \text{ point in the same direction}$$

in this case $\lambda \in \mathbb{R}$

$$\left[\frac{\partial J}{\partial u_1} \; \frac{\partial J}{\partial u_2}\right] = -\lambda \left[\frac{\partial g}{\partial u_1} \; \frac{\partial g}{\partial u_2}\right]$$



more constraints

$$\underbrace{\left[\frac{\partial J}{\partial u_1} \cdots \frac{\partial J}{\partial u_n}\right]}_{\frac{\partial J}{\partial u} \in \text{ span rows of } \frac{\partial g}{\partial u}} = -[\lambda_1 \, \lambda_2] \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_n} \\ \hline \frac{\partial g_2}{\partial u_1} & \cdots & \frac{\partial g_2}{\partial u_n} \end{bmatrix}}_{\frac{\partial g}{\partial u}}$$

→ taking lin comb of rows

Dynamic Programming: optimization in time

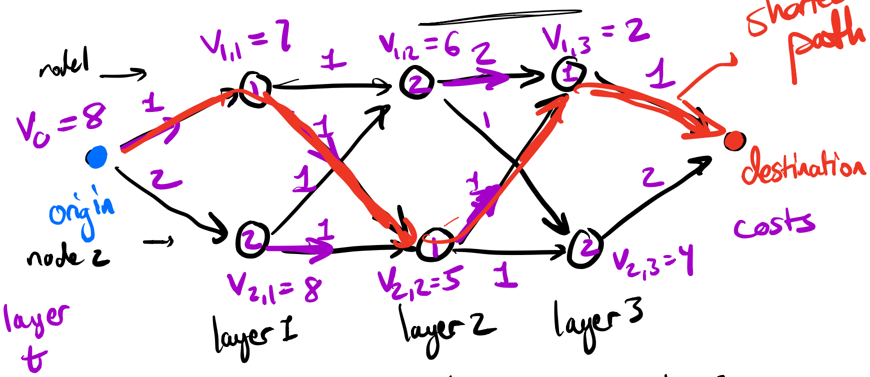
$$\min_u \sum_{t=0}^{T} l(u,t)$$

- key insight: figure out what to do at time $t = T$
  use that to inform decision at $t = T-1$
  and so on stepping backwards

"optimal cost current time" = "immediate cost" + "optimal cost-to-go next step"

Shortest Path

$$\min_{\text{path}} \sum_{e \in \text{path}} l_e$$



$$V^*_{x,t} = \begin{pmatrix}\text{opt} \\ \text{cost} \\ \text{to go}\end{pmatrix} \text{ node } x, \text{ layer } t$$

(x ↓ node, t ↓ layer)

analogy: layers are times

$$V_{1,2} = \min\left\{2 + 2 + \underbrace{V_{1,3}}_{2},\ 2 + 1 + \underbrace{V_{2,3}}_{4}\right\}$$

6 7

Analogy
layers → time
nodes → states
edges → controls

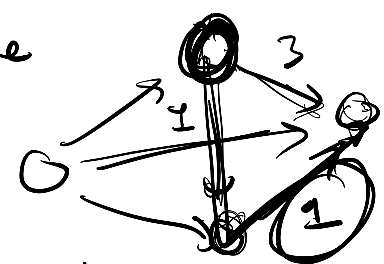
$$V_{x,t} = l_x + \gamma \min_{x' \in N_x}\left(l_{x,x'} + V_{x',t+1}\right) \Leftarrow$$

Bellman eqn.

node, layer, cost at node, nodes we can get to from x

Sometimes use discount factor $0 < \gamma \le 1$

iterate over values till V_x converges $\forall x$



OPTIMAL CONTROL: Dynamic Programming for Control

layers → time, nodes → state space, edges → controls

discrete, continuum, discrete, continuum

Continuous Time

$$\min_u \int_0^T \underbrace{l(x,u,t)}_{\text{immediate cost}}\,dt + \underbrace{l(x(T),T)}_{\text{terminal cost}}$$

s.t. $\dot{x} = f(x,u,t)$, $x(0) = x_0$

OPT - COST-TO-GO $\underbrace{V_t(x)}_{\text{function of state } x}$

Hamilton-Jacobi Bellman Eqn

$$-\dot{V}_t(x) = \min_{u(t)} l(x(t),u(t),t) + \frac{\partial V}{\partial x} f(x,u,t)$$

with $V_T(x) = l(x(T),T)$

Discrete Time:

$$\min_u \sum_{t=0}^{T-1} l(x[t],u[t],t) + l(x[T],T)$$

s.t. $x[t+1] = f(x[t],u[t],t)$, $x[0] = x_0$

OPT-COST-TO-GO $V_t(x)$

$$V_t(x) = \min_{u[t]}\left(l(x[t],u[t],t) + V_{t+1}(x[t+1])\right)$$

$$V_{t+1}(f(x[t],u[t],t))$$

$V_T(x) = l(x[T],T)$

Backwards partial differential eqn.

$$\cancel{V_t} = \min_{u(t)} \ell\Delta t + \underbrace{V_{t+\Delta t}(x+\Delta x)}_{\cancel{V_t(x)} + \frac{\partial V}{\partial t}(x)\Delta t + \frac{\partial V}{\partial x}\Delta x}$$

$$-\frac{\partial V}{\partial t}\Delta t = \min_{u(t)} \ell\Delta t + \frac{\partial V}{\partial x}\Delta t\, f$$

$$-\frac{\partial V}{\partial t} = \min_{u} \ell + \frac{\partial V}{\partial x} f$$

LINEAR QUADRATIC REGULATOR:

$$\min_{u:[0,T]\to\mathbb{R}^m} \int_0^T x^TQ(t)x + u^TR(t)u\,dt + x(T)^TQ_Tx(T)$$

s.t. $\dot{x} = A(t)x + B(t)u$, $x(0) = x_0$

HAMILTON JACOBI BELLMAN

$$-\dot{V} = x^TQ(t)x + \min_{u(t)}\left(u(t)^TR(t)u(t) + \frac{\partial V}{\partial x}\big(A(t)x + B(t)u\big)\right)$$

with $V_T(x) = x^TQ_Tx$

$$Q(t) = Q(t)^T \geq 0$$

$$R(t) = R(t)^T > 0$$

often:

$$u^TRu = u^T\begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix}u = \sum_i r_i u_i^2$$

BACKWARDS ITERATIVE EQN: use to solve for $V_t(x)$

$$\min_{u} \sum_{t=0}^{T-1} x[t]^TQ[t]x[t] + u[t]^TR[t]u[t] + x[T]^TQ[T]x[T]$$

s.t. $x[t+1] = A[t]x[t] + B[t]u[t]$, $x[0] = x_0$

$$V_t(x) = x^TQ[t]x + \min_{u[t]}\left(u[t]^TR[t]u[t] + V_{t+1}[x[t+1]]\right)$$

with $V_T(x) = x^TQ[T]x$

$$Q[t] = Q[t]^T \geq 0$$

$$R[t] = R[t]^T > 0$$

often: want to minimize $y = Cx$ → $Q = C^TC$

$C = [\underbrace{\quad\quad}_{\text{fat}}]$

Q is not pos def.

$\Rightarrow (A, C)$ observable

$V_t(x) = x(t)^T P(t)\, x(t)$

Initialize $V_T(x) = x^T Q[T]\, x$

what is the form of $V_t(x)$?

TOSHOW: if $V_{t+1}(x) = x^T P(t+1)\, x \rightarrow V_t = x^T P(t)\, x$

DISCRETE TIME: for $x[t] \rightarrow x$ $\quad u[t] = u$ $\quad (x[t+1] - \bar{x}[t+1])^T$ tracking

$$V_t(x) = \underline{x^T Q[t] x} + \min_u \left(\underline{u^T R[t] u} + \underline{x[t+1]^T P[t+1]\, x[t+1]}\right)$$

$u - \bar{u}$ $\quad (A[t]x + B[t]u)^T$

$$V_t(x) = x^T Q[t] x + \min_u \Big(u^T \underline{(R[t] + B[t]^T P[t+1] B[t])}\, u \;①\; +$$

$$\underline{2 x^T A[t]^T P[t+1] B[t]\, u} + ②$$

$$x^T A[t]^T P[t+1] A[t]\, x \Big)$$

optimization:

$u - \bar{u}$ $\quad x - \bar{x}$ $\quad u - \bar{u}$

$$\frac{\partial}{\partial u} = 0 \quad \Rightarrow \quad 2 \underline{u^T R[t]} + 2(\underline{A[t]\, x} + \underline{B[t]\, u})^T P[t+1]\, B = 0$$

solving for u:

$$\boxed{u^*[t] = \underline{K[t]}\, \underline{x[t]} = -\underbrace{(R[t] + B[t]^T P[t+1] B[t])^{-1} B[t]^T P[t+1] A[t]}_{K[t]}\, x[t]}$$

$x - \bar{x}$ $\quad x - \bar{x}$

plugging in u^* ... (unless noted everything is a func of t

$$V_t(x) = x^T \Big(Q + A^T P[t+1] A + \underline{A P[t+1] B (R + B^T P[t+1] B) B^T P[t+1] A} \;①$$

$$\underline{- 2 A P[t+1] B (R + B^T P[t+1] B) B^T P[t+1] A} \;② \Big)\, x$$

$$V_t(x) = x^T\Big(\underbrace{Q + A^T P[t+1] A - A P[t+1] B (R + B^T P[t+1] B) B^T P[t+1] A}_{P[t]}\Big) x$$

$$\boxed{\begin{aligned} P[t] &= Q[t] + A[t]^T P[t+1] A[t] - A[t] P[t+1] B[t] \big(R[t] + B[t]^T P[t+1] B[t]\big) \times \\ &\qquad \big(B[t]^T P[t+1] A[t]\big) \\ P[t] &\leftarrow \text{takes } P[t+1] \\ \text{initialize at} \quad P[T] &= Q[T] \end{aligned}}$$

Riccati Eq^n

Continuous Time:

(again everything is a function of time)

$$V_t(x) \approx \min_u \Delta t\, x^T Q x + \Delta t\, u^T R u + V_{t+\Delta t}(x + \Delta t(Ax + Bu))$$

$$\min_u \Delta t\, x^T Q x + \Delta t\, u^T R u + (x + \Delta t(Ax+Bu))^T P(t+\Delta t)(x + \Delta t(Ax+Bu))$$

$$\min_u \Delta t\, x^T Q x + \Delta t\, u^T R u + (x + \Delta t(Ax+Bu))^T (P(t) + \Delta t \dot{P})(x + \Delta t(Ax+Bu))$$

$$\cancel{x^T P x}^{\,0} = \cancel{x^T P(t) x}^{\,0} + \min_u \Delta t\, x^T Q x + \Delta t\, u^T R u + (x + \Delta t(Ax+Bu))^T (P(t) + \Delta t \dot{P})(x + \Delta t(Ax+Bu))$$

$$0 = \Delta t\, (Ax+Bu)^T P(t) x + x^T P(t)(Ax+Bu)\Delta t + \Delta t\, x^T \dot{P}(t) x + \text{h.o.t.}$$

$$\frac{\partial}{\partial u} = 0 \qquad \Delta t\, 2u^T R + \Delta t\, 2x^T P(t) B = 0$$

$$\Rightarrow \boxed{u^* = -R(t)^{-1} B(t)^T P(t)\, x(t) = K(t)\, x(t)}$$

plugging in u^*

$$-x^T\dot{P}x = x^T\Big(Q + \underbrace{K^TRK}_{PBR^{-1}B^TP} + A^TP(t) + P(t)A + \underbrace{K^TB^TP(t) + P(t)BK}_{-2P(t)BR^{-1}B^TP(t)}\Big)x$$

$$\boxed{\begin{aligned} -\dot{P}(t) &= Q(t) + A(t)^TP(t) + P(t)A(t) - P(t)B(t)R(t)^{-1}B(t)^TP(t) \\ P(T) &= Q_T \end{aligned}}$$

LINEAR QUADRATIC REGULATOR:

$$\min_{u:[0,T]\to\mathbb{R}^m} \int_0^T x^TQ(t)x + u^TR(t)u\,dt + x(T)^TQ_Tx(T)$$

s.t. $\dot{x} = A(t)x + B(t)u$, $x(0) = x_0$

$$\min_u \sum_{t=0}^{T-1} x[t]^TQ[t]x[t] + u[t]^TR[t]u[t] + x[T]^TQ[T]x[T]$$

s.t. $x[t+1] = A[t]x[t] + B[t]u[t]$, $x[0] = x_0$

HAMILTON JACOBI BELLMAN

$$-\dot{V} = x^TQ(t)x + \min_{u(t)}\left(u(t)^TR(t)u(t) + \frac{\partial V}{\partial x}\big(A(t)x + B(t)u\big)\right)$$

with $V_T(x) = x^TQ_Tx$

$$V_t(x) = x^TQ[t]x + \min_{u[t]}\Big(u[t]^TR[t]u[t] + V_{t+1}[x[t+1]]\Big)$$

with $V_T(x) = x^TQ[T]x$

Solution:

CONTROL: $u^* = -R(t)^{-1}B(t)^TP(t)x(t) = K(t)x(t)$

COST-TO-GO: $x(t)^TP(t)x(t)$ where

$$-\dot{P}(t) = Q(t) + A(t)^TP(t) + P(t)A(t) - P(t)B(t)R(t)^{-1}B(t)^TP(t)$$

with: $P(T) = Q_T$

RICCATI DIFF EQ

CONTROL: $u^*[t] = -(R[t] + B[t]^TP[t+1]B[t])^{-1}B[t]^TP[t+1]A[t]\,x[t] = K[t]\,x[t]$

COST-TO-GO: $x[t]^TP[t]x[t]$

$$P[t] = Q[t] + A[t]^TP[t+1]A[t] - A[t]^TP[t+1]B[t]\big(R[t] + B[t]^TP[t+1]B[t]\big)\big(B[t]^TP[t+1]A[t]\big)$$

with: $P[T] = Q[T]$

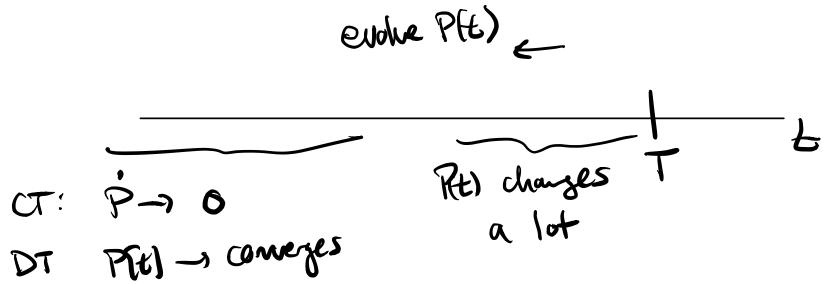
RICCATI DIFFERENCE EQN

Interpretation $\underline{V_t(x)} = \underbrace{x^T P(t)\, x}_{\text{this form is because LQR}}$ = "opt cost to go from time t to T"

For LTI System:

$0 \ll T$

$\downarrow$



CONTINUOUS TIME

$\dot{P} = 0$

$$\boxed{\begin{aligned} u &= Kx = -R^{-1}B^T P x \\ -\dot{P} &= 0 = A^T P + PA + Q - PBR^{-1}B^T P \end{aligned}}$$

single P: infinite horizon or steady solution

Algebraic Riccati Eqn

DISCRETE TIME:

$$\boxed{\begin{aligned} u &= Kx = -(R + B^T P B)^{-1} B^T P A \\ P &= Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A \end{aligned}}$$

DISC ALGEBRAIC RICCATI EQN

Note: this requires that A, B, Q, R are fixed in time cause otherwise P wouldn't converge

Extensions:

cross term: $\int_0^T x^T Q(t) x + 2x^T N(t) u + u^T R(t) u \, dt$

same derivation...

well posed based $\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \rightarrow$ positive semi definite

tracking problem $\rightarrow$ $\int_0^T (x-\bar{x})^T Q(t)(x-\bar{x}) + (u-\bar{u})^T R(t)(u-\bar{u}) \, dt$

To derive: basically just plug in $x-\bar{x}$ for x

$u-\bar{u}$ for u

Nonlinear traj. tracking: nominal control $\bar{u}$

$\rightarrow$ nominal state traj $\bar{x}$

linearize around $\bar{x}, \bar{u}$ $\rightarrow$ $A(t) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{u}}$

SOLVE LTV LQR tracking problem $\quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{\bar{x},\bar{u}}$

$u = \bar{u} + K(t)(x-\bar{x})$

Derivation from Lagrange Multiplier Perspective

$$\min_{\substack{u:[0,T]\to\mathbb{R}^n \\ x:[0,T]\to\mathbb{R}^n}} J = \int_0^T x^T Q(\tau) x + u^T R(t) u \, d\tau + x^T Q(T) x$$

s.t. $\dot{x} = A(t)x + B(t)u$, $x(0) = x_0$

Lagrange Multiplier: $\lambda : [0,T] \to \mathbb{R}^n$ — enforces the dynamics

Lagrangian: $L = J + \int_0^T \lambda(\tau)^T (A(\tau)x(\tau) + B(\tau)u(\tau) - \dot{x}(\tau)) d\tau$

signal

dynamics

constraint

$0 = -\dot{x}(\tau) + A(\tau)x(\tau) + B(\tau)u(\tau)$

inner product for signals

$\underbrace{\lambda : [0,T] \to \mathbb{R}^n}_{\text{signal}}$

$\lambda^T g(u)$

Optimality Conditions:

$\underbrace{\frac{\partial L}{\partial u}(t) = 0}_{\text{signals}}$ $\quad \underbrace{\frac{\partial L}{\partial x}(t) = 0}_{\text{signals}}$ $\quad \underbrace{\frac{\partial L}{\partial \lambda}(t) = 0}_{\text{signals}}$ must hold for $t \in [0,T]$

meaning of $\frac{\partial L}{\partial u}(t)$ is that $\Delta L = \int_0^T \frac{\partial L}{\partial u}(\tau) \Delta u(\tau) d\tau$

similarly for $\frac{\partial L}{\partial x}(t)$ and $\frac{\partial L}{\partial \lambda}(t)$ ...

$$\frac{\partial L}{\partial u}(t):\ u(t)^T R(t) + \lambda(t)^T B(t) = 0$$

$$\Rightarrow \boxed{u(t) = -R(t)^{-1} B^T \lambda(t)}$$ optimal control law

$$\int_0^T \lambda(\tau)^T \dot{x}(\tau)\,d\tau = \lambda(T)^T x(T) - \lambda(0)^T x(0) - \int_0^T \dot{\lambda}(\tau)^T x(\tau)\,d\tau$$

(integration by parts ...)

$\lambda^T B R^{-1}$

$$\frac{\partial L}{\partial x}(t) = x(t)^T Q(t) + \lambda(t)^T A(t) + \dot{\lambda}(t)^T = 0$$

costate equation

$$\Rightarrow \boxed{-\dot{\lambda}(t)^T = \lambda(t)^T A(t) + x(t)^T Q(t) \qquad \lambda(T)^T = x(T)^T Q(T)}$$

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$$\frac{\partial L}{\partial \lambda}(t) = \boxed{\dot{x}(t) - A(t)x(t) - B(t)u(t) = 0}$$ dynamics

State equation

$$\boxed{\lambda(t)^T = x(t)^T P(t)}$$

can solve for $x(t), \lambda(t)$

Hamiltonian System:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B(t)^T \\ -Q(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

with $\underline{x(0) = x_0}$ $\quad \underline{\lambda(T) = Q(T)x(T)}$

2 point boundary value problem.

$\lambda(t)$: tracks the gradient of the cost-to-go

$x(t)$: tracks the state