KALMAN FILTER:

- DUAL PROBLEM TO LQR

  Similar to the relationship feedback design (LQR) & observer design (KALMAN FILTER)

DISCRETE TIME:

(use $A_t = A[t]$, etc...)

independent of state and v_t

Dynamics: $x_{t+1} = A_t x_t + B_t u_t + w_t$

$y_t = C_t x_t + v_t$

independent of state and w_t

$$\underbrace{\begin{matrix} w_t \sim N(0, \underline{W_t}) \\ v_t \sim N(0, \underline{V_t}) \end{matrix}}_{\text{Multivariate Gaussian}}$$

Estimate of state:

estimate: $\hat{x}_t$

error: $e_t = \hat{x}_t - x_t$

error covariance: $S_t = E[(\hat{x}_t - x_t)(\hat{x}_t - x_t)^T]$

want evolve both $\hat{x}_t$ & S_t

$\hat{x}_t$: estimate of state

S_t: what we expect the error in our estimate to be

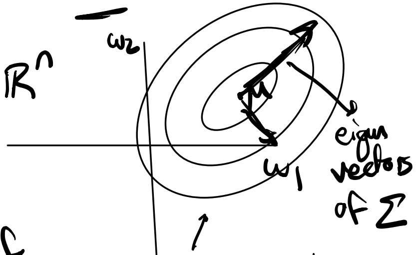
Kalman filter basically uses the information in S_t to update $\hat{x}_t$ to minimize $\hat{x}_t - x_t$

mean μ: $\mu = E[w]$

cov $\underline{\Sigma}$: $E[(w-\mu)(w-\mu)^T]$

symmetric

$w \in \mathbb{R}^n$

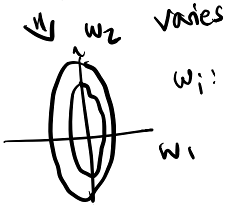


pdf

$$f(w) = \frac{\det(\Sigma)^{-1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{2}(w-\mu)^T \Sigma^{-1} (w-\mu)}$$

if Σ diagonal: each element of w varies separately

w_i: independent



off diagonal terms: ij covariance between w_i & w_j

Note: updating covariance matrices...

$\text{cov}(e) = S$ $\quad$ $\text{cov}(Me) = MSM^T$

$\downarrow$

$E[ee^T] = S$ $\quad$ $E[Mee^TM^T] = ME[ee^T]M^T$

know covariance: then $E[\sum_i (w_i - M_i)^2] = TR(\Sigma)$

Steps in KF:

true state evolves as : $\underline{x_{t+1}} = A_t x_t + B_t u_t + \underline{w_t}$

noise in the interval $[t, t+1]$

① Predict

$$\hat{x}'_{t+1} = A_t \hat{x}_t + B_t u_t$$

$$S'_{t+1} = E\left((\hat{x}'_{t+1} - x_{t+1})(\hat{x}'_{t+1} - x_{t+1})^T\right)$$

$$= E\left((A(\hat{x}_t - x_t) + w_t)(A(\hat{x}_t - x_t) + w_t)^T\right)$$

$$= A\,E\left((\hat{x}_t - x_t)(\hat{x}_t - x_t)^T\right)A^T$$

$$+ A\,\overset{0}{\cancel{E\left((\hat{x}_t - x_t)w_t^T\right)}} + \overset{0}{\cancel{E\left(w_t(\hat{x}_t - x_t)^T\right)}}A^T$$

$$+ \quad E(w_t w_t^T)$$

noise between time t and t+1 → independent of x_t

$$= \underline{A S_t A^T} + W_t \rightarrow$$

+ noise from dynamics

error propagation through dynamics

Note w & u are independent

$\Rightarrow E[wu^T] = E[w]E[u]^T$

② Update

innovation

$$\hat{x}_{t+1} = \hat{x}'_{t+1} + \boxed{L_{t+1}}\overbrace{(C_{t+1}\hat{x}'_{t+1} - y_{t+1})}^{\text{innovation}}$$

$$S_{t+1} = E[(\hat{x}_{t+1} - x_{t+1})(\hat{x}_{t+1} - x_{t+1})^T]$$

$$= E\left[(\hat{x}'_{t+1} - x_{t+1} + L_{t+1}C_{t+1}(\hat{x}'_{t+1} - x_{t+1}) + L_{t+1}v_{t+1})(\hat{x}'_{t+1} - x_{t+1} + L_{t+1}C_{t+1}(\hat{x}'_{t+1} - x_{t+1}) + L_{t+1}v_{t+1})^T\right]$$

Let $z = \hat{x}'_{t+1} - x_{t+1}$

$$= E[zz^T] + E[zz^T]C_{t+1}^T L_{t+1}^T + \overset{0}{\cancel{E[zv_{t+1}^T]}}L_{t+1}^T$$

$$L_{t+1}C_{t+1}E[zz^T] + L_{t+1}C_{t+1}E[zz^T]C_{t+1}^T L_{t+1}^T + L_{t+1}C_{t+1}\overset{0}{\cancel{E[zv_{t+1}^T]}}L_{t+1}^T$$

$$L_{t+1}\overset{0}{\cancel{E[v_{t+1}z^T]}} + L_{t+1}\overset{0}{\cancel{E[v_{t+1}z^T]}}C_{t+1}^T L_{t+1}^T + L_{t+1}E[v_{t+1}v_{t+1}^T]L_{t+1}$$

Independence of v_{t+1} & z

Typo from below... CORRECTED VERSION →

$$\star \; S_{t+1} = \underline{S'_{t+1} + L_{t+1}C_{t+1}S'_{t+1} + S'_{t+1}C^T_{t+1}L^T_{t+1} + L_{t+1}(C_{t+1}S'_{t+1}C^T_{t+1}+V_{t+1})L^T_{t+1}}$$

(again making the assumption that V_{t+1} is independent of $\hat{x}_{t+1}$ and x_{t+1})

How do we pick L_{t+1}:

choose L_{t+1} to minimize $\sum_i [(\hat{x}_{t+1})_i - (x_{t+1})_i]^2 = Tr(S_{t+1})$

$$\min_{L_{t+1}} \; Tr(S_{t+1})$$

$$\frac{\partial}{\partial L_{t+1}} Tr(\underline{S^{\star}_{t+1}}) = 0 : \quad 2S'_{t+1}C^T_{t+1} + 2L_{t+1}(C_{t+1}S'_{t+1}C^T_{t+1}+V_{t+1}) = 0$$

$$\Rightarrow \boxed{L_{t+1} = -S'_{t+1}C^T_{t+1}(C_{t+1}S'_{t+1}C^T_{t+1}+V_{t+1})^{-1}}$$

Kalman gain eqn:

Substituting in to covariance update:

$$S_{t+1} = S'_{t+1} + L_{t+1}C_{t+1}S'_{t+1} + S'_{t+1}C^T_{t+1}L^T_{t+1} + L_{t+1}(C_{t+1}S'_{t+1}C^T_{t+1}+V_{t+1})L^T_{t+1} \leftarrow$$ corrected

$$= S'_{t+1} + L_{t+1}C_{t+1}S'_{t+1}$$

$$= (I + L_{t+1}C_{t+1})S'_{t+1}$$

Summary:

	Predict	update:	Initialization
$\hat{x}$:	$\hat{x}'_{t+1} = A_t\hat{x}_t + B_t u_t$	$\hat{x}_{t+1} = \hat{x}'_{t+1} + L_{t+1}(C_{t+1}\hat{x}'_{t+1} - y_{t+1})$	$\hat{x}_0 = E[x_0]$
S:	$S'_{t+1} = A_tS_tA_t^T + W_t$	$S_{t+1} = (I + L_{t+1}C_{t+1})S'_{t+1}$	$S_0 = E[x_0x_0^T]$

where: $L = -S'_{t+1}C^T_{t+1}(C_{t+1}S'_{t+1}C^T_{t+1} + V_{t+1})^{-1}$

Extended Kalman Filter: nonlinear systems.

$$x_{t+1} = f(x_t, u_t, t)$$

$$y_t = h(x_t)$$

set $A_t = \left.\frac{\partial f}{\partial x}\right|_{x_t, u_t}$ $\quad B_t = \left.\frac{\partial f}{\partial u_t}\right|_{x_t, u_t}$ $\quad C_t = \left.\frac{\partial h}{\partial x}\right|_{x_t, u_t}$

Summary:

Predict

$\hat{x}$: $\rightarrow \hat{x}'_{t+1} = \underline{f(x_t, u_t, t)}$

S: $S'_{t+1} = A_t S_t A_t^T + W_t$

update:

nonlinear

$\hat{x}_{t+1} = \hat{x}'_{t+1} + L_{t+1}(\underline{h(x_{t+1})} - y_{t+1})$

$S_{t+1} = (I + L_{t+1} C_{t+1}) S'_{t+1}$

Initialization

$\hat{x}_0 = E[x_0]$

$S_0 = E[x_0 x_0^T]$

linearization

where: $L = -S'_{t+1} C_{t+1}^T (C_{t+1} S'_{t+1} C_{t+1}^T + V_{t+1})^{-1}$

LQG: Linear Quadratic Gaussian

Discrete time:

$$\min_u E\left[\sum_{t=0}^{T-1} x_t^T Q_t x_t + u_t^T R_t u_t\right] + x_T^T Q_T x_T$$

s.t. $x_{t+1} = A_{t+1} x_t + B_t u_t \qquad x_0 = x_0$

Solution:

$\hat{x}_k^+ = \hat{x}_k^- + K_k[y_k - H_k \hat{x}_k^-] \longleftrightarrow k+1$

$$\left[\hat{x}_{t+1} = A_t \hat{x}_t + B_t u_t + L_{t+1}(C_{t+1}(A_t \hat{x}_t + B_t u_t) - y_{t+1}), \quad \hat{x}_0 = E[x_0]\right]$$

$$u_t = +K_t \hat{x}_t$$

$P_k^+ = (I - K_k H_k) P_k^-$

observer:

$$L_t = -\underline{S_t C_t^T}(\underline{C_t S_t C_t^T + V_t})^{-1}$$

$$S_{t+1} = \underline{A_t} S_t A_t^T - A_t S_t \underline{C_t}^T(\underline{C_t} S_t \underline{C_t}^T + V_t)^{-1} C_t S_t A_t^T + W_t, \quad S_0 = E[x_0 x_0^T]$$

gain:

$$K_t = -(\underline{B_t^T P_{t+1} B_t + R_t})^{-1} \underline{B_t^T P_{t+1} A_t}$$

$$P_t = \underline{A_t^T} P_{t+1} A_t - A_t P_{t+1} \underline{B_t}(\underline{B_t^T} P_{t+1} \underline{B_t} + R_t)^{-1} B_t^T P_{t+1} A_t + Q_t, \quad P_T = Q_T$$

Relationship between LQR & KF

	LQR	KF
Riccati	Backwards from $P_T = Q_T$ " propogating the cost-to-go backwards "	Forwards from $S_0 = E[x_0 x_0^T]$ " propogating error covariance forward "
state cost/noise	Q_t state cost	W_t state noise
input/output	R_t input cost	V_t output noise
	P_t cost-to-go	S_t error covariance

Continuous Time LQG:

$$\min_u \quad E\left[\int_0^T x(t)^T Q(t)\, x(t) + u(t)^T R(t)\, u(t)\, dt + x(T)^T Q(T)\, x(T) \right.$$

$$\text{s.t.} \quad \dot{x} = A(t)x(t) + B(t)u(t) \qquad x(0) = x_0$$

Solution:

$$\dot{\hat{x}} = A(t)\hat{x}(t) + B(t)u(t) + L(t)\left(C(t)\hat{x}(t) - y(t)\right), \quad \hat{x}(0) = E[x_0]$$

$$u(t) = K(t)\hat{x}(t)$$

Where $L(t) = -S(t)C(t)^T V^{-1}(t)$

$$\dot{S}(t) = A(t)S(t) + S(t)A^T(t) - S(t)C^T(t)V^{-1}(t)C(t)S(t), \quad S(0) = E[x_0 x_0^T]$$

$$K(t) = -R^{-1}(t)B^T(t)P(t)$$

$$-\dot{P}(t) = A(t)^T P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B'(t)P(t) + Q(t), \quad P(T) = Q(T)$$

INFINITE HORIZON KALMAN FILTER:

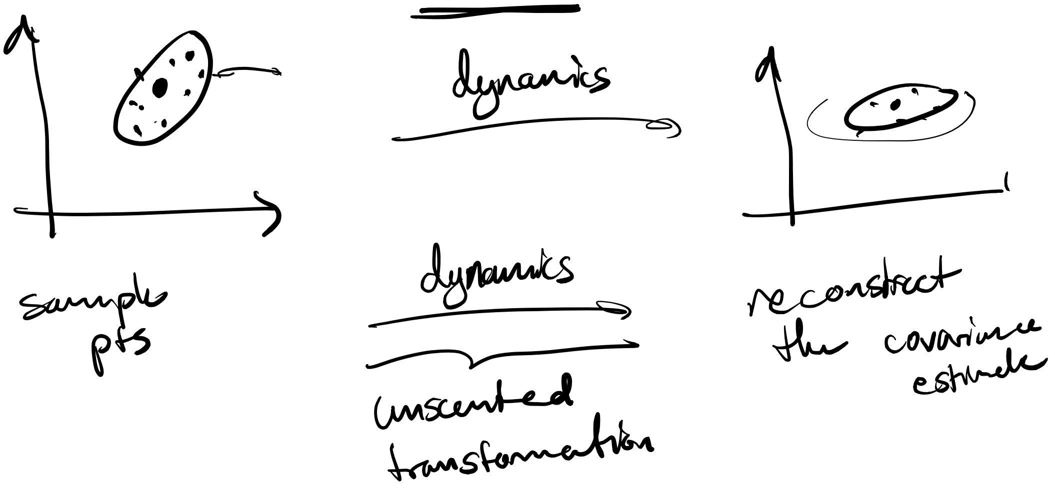
LTI systems with long time horizon $\dot{S} \to 0$

Algebraic Riccati Eqn:
$$0 = AS + SA^T - SC^TV^{-1}CS + W$$
$$L = -SC^TV^{-1}$$

LQG: First pass at design
tune after that for robustness.

other versions of KF:

- unscented KF → uses more sophisticated methods than Extended KF for propogating covariance (nonlinear covariance updated)



unscented KF : type of particle filter

Integral Feedback

$\dot{x} = Ax + Bu$ want y to track some reference r

$y = Cx$

want to penalize $Cx - r$ over time

creating a new state

$z = \int Cx - r \Rightarrow \dot{z} = Cx - r$

put a gain on z state forces $Cx = r$ as $t \rightarrow$

Augmented dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{bmatrix} B & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} u \\ r \end{bmatrix}$$

Example: state space PID control (want x to track 0

2nd order system

$$\begin{bmatrix} \dot{z} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & * & * \end{bmatrix} \begin{bmatrix} z \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u$$

$u = Kx = [K_I \; K_P \; K_D] \begin{bmatrix} z \\ x \\ \dot{x} \end{bmatrix}$

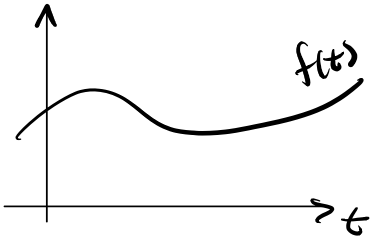
add another state $z = \int x$ or $\dot{z} = x$

FREQUENCY DOMAIN:

Laplace Transform:

Different way to represent time signals

functions of time → functions of frequency



$f(t)$: infinite dimensional vector

t : index

$f(t)$: value

"continuous index"

finite dim

i : index

x_i : value

What if we want $f(t)$ in a different basis of functions?

ex: represent $f(t)$ in polynomial basis

want to represent functions in a functional basis that makes taking derivatives easy...

Derivative: linear operator

$$\frac{d}{dt}(\alpha f(t) + \beta g(t)) = \alpha \frac{d}{dt} f(t) + \beta \frac{d}{dt} g(t)$$

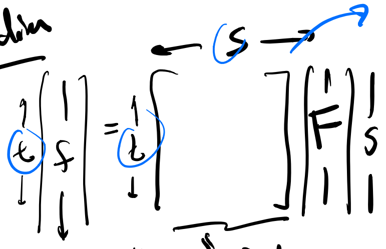
What if we want to represent $f(t)$ in an basis of eigenfunctions for the derivative operator?

Finite

$$X = T z$$

cols ↓ of T are eigenvectors of $\boxed{A}$

Inf dim



$\cancel{\mathbb{R}^{\infty \times \infty}}$

$\mathbb{R}^{[0,\infty)\times[0,\infty)}$

"cols" of this matrix are "eigenfunctions" of $\boxed{\frac{d}{dt}}$

Eigenfunctions of $\frac{d}{dt}$: $$\underbrace{\frac{d}{dt}}_{\text{operator}} e^{st} = \underbrace{s}_{\text{eigen value}}\underbrace{e^{st}}_{\text{eigen function}}$$

s can be whatever

want to write f(t) in eigenbasis

$$f(t) = \underbrace{\begin{array}{c|c} & \leftarrow s \rightarrow \\ \hline t & \left[\begin{array}{c} | \\ e^{st} \\ | \end{array}\right] \end{array}}_{} F(s)$$

coefficients of exponential functions

-1 inverse of the Laplace transform

$$F(s) = \begin{array}{c} | \\ s \\ | \end{array} \left[\begin{array}{ccccc} | & | & e^{st} & | & | \\ | & | & | & | & | \end{array}\right]^{-1} f(t)$$

not rigorous -- (just intuition) ↑

$\int g(t) f(t)\, dt$

"dot" product" of two functions

Fin dim: $\sum_i x_i y_i$

Inf dim: $\int_t f(t) g(t)\, dt$

inner product of e^{-st} & $f(t)$

Just intuition ↑

DON'T NEED TO KNOW

Laplace Transform

$$\boxed{\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\, dt}$$

need f(t) to grow slower than e^{st}

$$\mathcal{L}(\dot{f}(t)) = \int_0^\infty \dot{f}(t) e^{-st}\, dt = \overset{0}{\cancel{f(\infty)}}\underset{\text{decays}}{e^{-s\infty}} - f(0)\overset{1}{\cancel{e^{-s0}}} - \underline{\int_0^\infty f(t)(\dot{e^{-st}})\, dt}$$

Integration by parts!

$$+ s\int_0^\infty f(t) e^{-st}\, dt$$

$$F(s) = \mathcal{L}(f(t))$$

$$\mathcal{L}(\dot{f}(t)) = sF(s) - f(0)$$

$$\underline{\mathcal{L}(e^{at})} = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{(s-a)}$$

Convolution: $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau \Big\} \rightarrow$ like $\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$

$\mathcal{L}(f*g) = F(s)G(s)$ where $F(s) = \mathcal{L}(f)$ $G(s) = \mathcal{L}(g) \Big\} \rightarrow$ where transfer functions come

$\downarrow$

Proof:

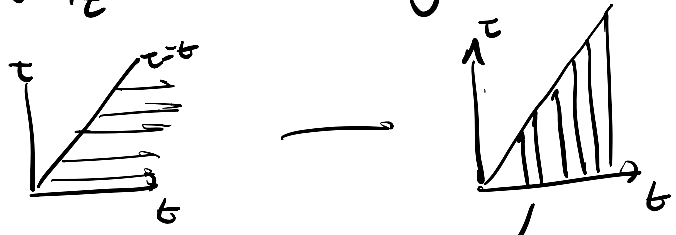
$$\left[\int_0^\infty f(\sigma)e^{-s\sigma}\,d\sigma\right]\left[\int_0^\infty g(\sigma)e^{-s\tau}\,d\tau\right]$$

$$\int_0^\infty \int_0^\infty f(\sigma)\,e^{-s(\sigma+\tau)}\,d\sigma\; g(\tau)\,d\tau$$

variable change $t = \sigma + \tau$

$$\int_0^\infty \int_\tau^\infty f(t-\tau)e^{-st}\,dt\; g(\tau)\,d\tau$$

Region of integration



$\downarrow$

$$F(s)G(s) = \underbrace{\int_0^\infty \int_0^t f(t-\tau)g(\tau)\,d\tau\, e^{-st}\,dt}$$

$$= \mathcal{L}(f*g)$$

$\downarrow$

$$\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$

"convolution of input & state response"

what is $\mathcal{L}(x(t))$: $\dot{x} = Ax + Bu$

$$sX(s) - x(0) = AX(s) + Bu(s)$$

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

compare $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$\mathcal{L}(y(t))$: $y(t) = Cx + Du$ ← general LTI output equation

$$Y(s) = CX(s) + DU(s)$$

denominator of transfer function

$$= \underbrace{C(sI-A)^{-1}x(0)}_{} + \Big[\underbrace{C\boxed{(sI-A)^{-1}}B + D}_{G(s)}\Big]u(s)$$

transient response decays if A is stable

if s is an eigenvalue then the transfer function blows up

$G(s)$ ↑ transfer function from inputs $u(t)$ to outputs $y(t)$

Single Input/Single Output: (SISO) $C(sI-A)^{-1}B$ ← scalar function of s

Multi Input / Multi output: (MIMO) $\underset{\downarrow}{o}\,[\;C\;]\,(sI-A)^{-1}\,[B]^{m}$ ← matrix of transfer functions

m → # of inputs

o → # of outputs

Design:

- stabilize plant through feedback $\underline{A+BK}$

  eigenvalues are in open left half of the complex plane

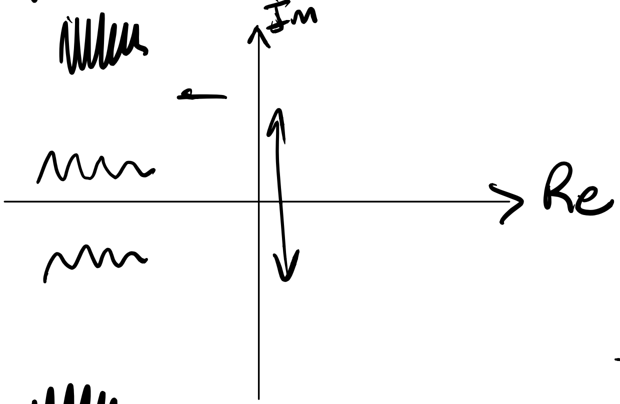
- also want to control the frequency response of the system

how the sys. responds to
high & low frequencies

Steady state behaviour: $Y(s) = G(s)U(s)$

Frequency Response of $G(s)$:

→ plugging in jw (or iw) → imaginary part of s.



if eigenvalues of $A+BK$ are in OLHP

→ transients decay

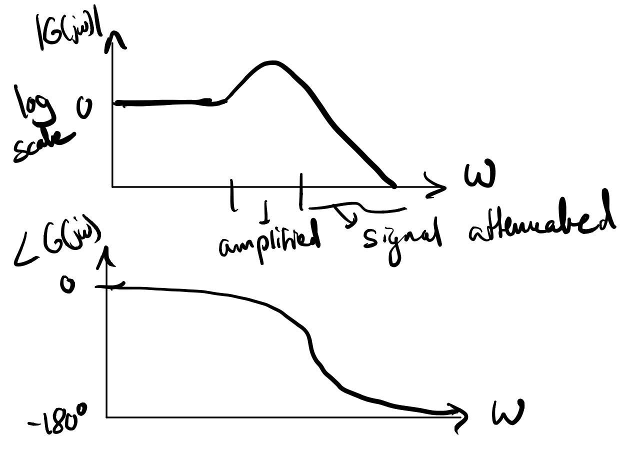
→ left w frequency response

Freq. Resp.

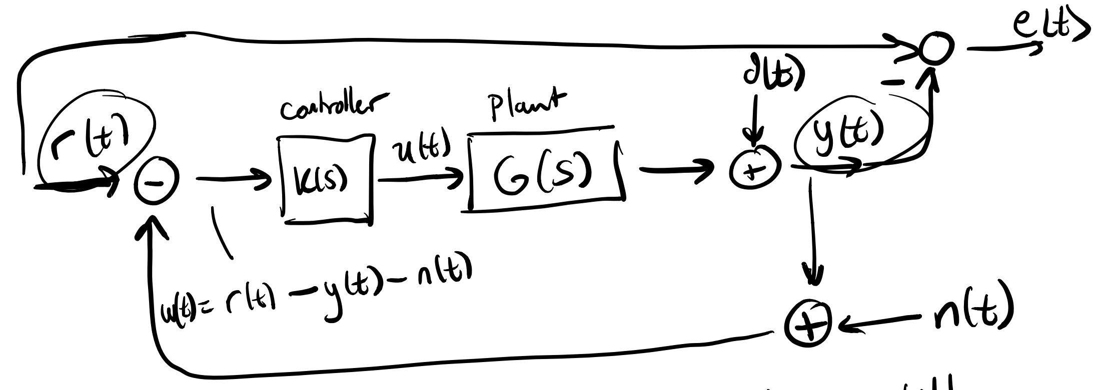
Amplitude: $|G(jw)|$ mag. of response at different frequencies

Phase: $\angle G(jw)$

Bode Plot:



CLOSED LOOP DIAGRAM:



Note : $G(s)K(s)$ is square & invertible

Transfer Functions :

$$Y(s) = D(s) + G(s)K(s)W(s)$$

$$W(s) = R(s) - Y(s) - N(s)$$

$$E(s) = R(s) - Y(s)$$

$$Y(s) = (I+GK)^{-1}GK(R-N) + (I+GK)^{-1}D$$

will double check.

$$E(s) = [I-(I+GK)^{-1}GK]R + [I+GK]^{-1}GKN - [I+GK]^{-1}D$$

some matrix algebra ↓
(matrix inversion lemma)

$$E(s) = (I+GK)^{-1}(R-D) + (I+GK)^{-1}GKN$$

matrix fact : $(I+GK)^{-1}GK = GK(I+GK)^{-1}$

$$\boxed{\begin{aligned} &\text{OUTPUT} \quad Y(s) = GK(I+GK)^{-1}(\overset{\text{ref.}}{R} - \overset{\text{noise}}{N}) + (I+GK)^{-1}\overset{\text{dist}}{\boxed{D}} \\ &\text{ERROR} \quad E(s) = (I+GK)^{-1}(\underset{\text{ref}}{R} - \underset{\text{dist}}{D}) + GK(I+GK)^{-1}\underset{\text{noise}}{N} \end{aligned}}$$

$S(s) = (I+GK)^{-1}$ system sensitivity

$T(s) = GK(I+GK)^{-1}$ complementary sensitivity

$$\left.\begin{aligned} Y(s) &= \underline{S(s)}\,D(s) + \underline{T(s)}\,(R(s) - \underline{N(s)}) \\ E(s) &= \underline{S(s)}\,(R(s) - D(s)) + \underline{T(s)}\,\underline{N(s)} \end{aligned}\right]$$

① Disturbance rejection (protect against $D(s)$): want $\underline{S(s)}$ to be small at frequencies where $\underline{R(s)}$ & $\underline{D(s)}$ are large

② Noise rejection (protect against $N(s)$): want $T(s)$ to be small at freq. when $N(s)$ is large

Unfortunately: $S + T = I \Leftarrow$ fixed

$$(I+GK)^{-1} + GK(I+GK)^{-1} = (I+GK)(I+GK)^{-1} = I$$

SISO

MIMO: look at the mag of singular values of $T(s)$ & $S(s)$