Transfer Functions

$$\xrightarrow{u(s)} \boxed{G(s)} \longrightarrow y(s) \qquad y(s) = G(s)u(s)$$

freq. functions
determined from the Laplace transfom

s is a complex #

$G(s)$: s in complex

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

$\underset{\text{Fourier Transform}}{\underline{G(j\omega)}}$: $j\omega$ is imag axis

Examples:

$\mathcal{L}(\delta(t)) = 1$ impulse: excites all frequencies

$\mathcal{L}(\dot{x}) = sX(s) - X(0)$ Differentiation

$\mathcal{L}\left(\int_0^t x\,dt\right) = \frac{1}{s} X(s)$ Integration

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$\mathcal{L}\left(\int_0^t e^{a(t-\tau)} b\, u(\tau) d\tau\right) = \underset{\sim}{\frac{b}{s-a}} \underline{U(s)}$ convolution

visualization

$G(s)$: stable linear system

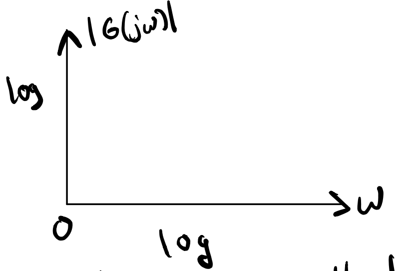
if $u(s) = u_0 \sin(\underline{\omega} t + \alpha)$

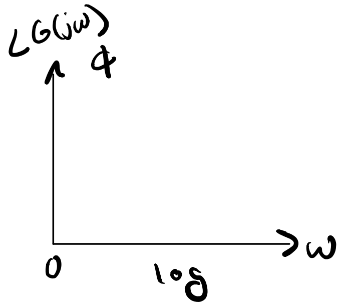
then $y(s) = y_0 \sin(\underline{\omega} t + \underline{\beta})$

$$\underset{\text{complex \#}}{\underline{G(j\omega)}} = \frac{|y_0|}{\underset{\text{mag}}{\underline{|u_0|}}} \underset{\text{phase}}{e^{j\phi}} \qquad |G(j\omega)| = \frac{|y_0|}{|u_0|}$$

$$\angle G(j\omega) = \phi = \beta - \alpha$$

Bode Plots:



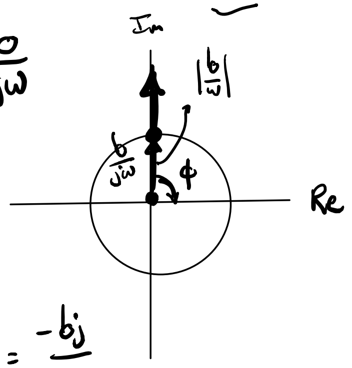


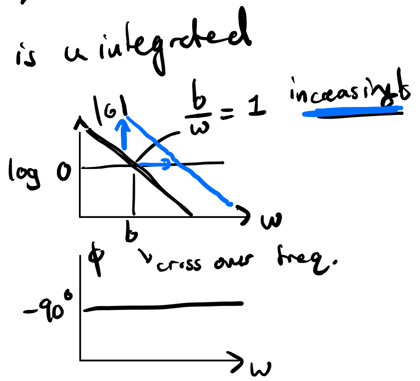
functions in matlab

Example: $\dot{x} = bu \quad y = x$

$y = x = \underset{\sim}{\frac{b}{s}} u \quad$ y is u integrated

$\frac{b}{j\omega}$





$$\frac{b}{j\omega} = \frac{-bj}{\omega}$$

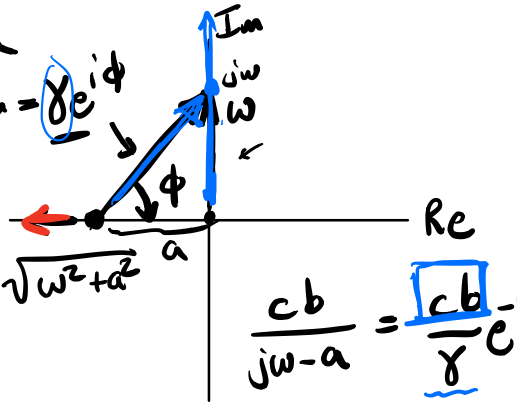
$$= \frac{b}{\omega}(-j) = \underset{\sim}{\frac{b}{\omega}} e^{-\frac{\pi}{2}j}$$

Example $\dot{x} = ax + bu \quad y = cx \quad$ a in LHP

$$y(s) = \frac{cb}{s-a}u(s) - \frac{c}{s-a}x(0)$$

$s-a$

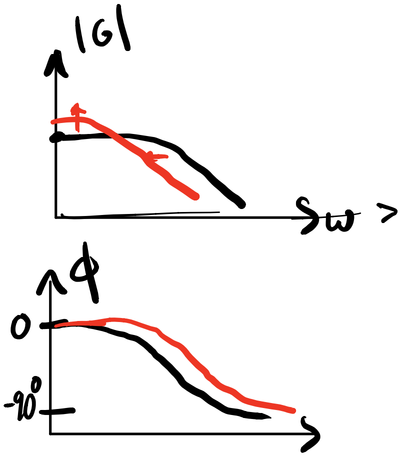
$j\omega - a = \gamma e^{i\phi}$



$\gamma = \sqrt{\omega^2 + a^2}$

ϕ

$$\frac{cb}{j\omega - a} = \frac{\boxed{cb}}{\underset{\sim}{\gamma}} e^{-i\phi}$$



physical interpretation:

slower frequencies: ω small → smaller phase lag.

faster frequencies: ω large → larger phase lag.

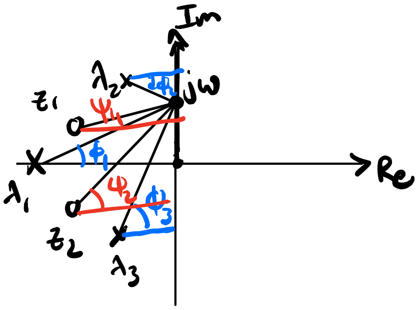
Example:

$$G(s) = \frac{(s-z_1)\cdots(s-z_m)}{(s-\lambda_1)\cdots(s-\lambda_n)}$$

} → for physical systems causal $m \le n$

= proper trans. func

< strict. proper trans.



$s-z_1 \cdot s-z_2$

$\delta_1 e^{j\psi_1} \quad \delta_2 e^{j\psi_2}$

$\underbrace{s-\lambda_1}_{\gamma_1 e^{j\phi_1}} \quad \underbrace{s-\lambda_2}_{\gamma_2 e^{j\phi_2}} \quad s-\lambda_3$

$\gamma_3 e^{j\phi_3}$

$$G(j\omega) = \frac{\delta_1\cdots\delta_m}{\gamma_1\cdots\gamma_n} e^{j\left(\sum_k \psi_k - \sum_k \phi_k\right)}$$

mag of terms in the numerator increase gain

mag of terms in the denominator decrease gain

phase of terms in the numerator increase phase

phase " " " " denominator decrease phase

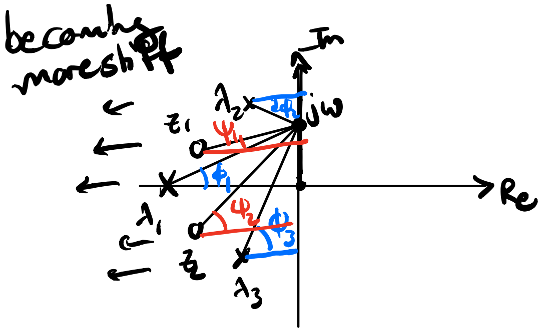
$$G(s) = \frac{(s-z_1)\cdots(s-z_m)}{(s-\lambda_1)\cdots(s-\lambda_n)}$$

→ roots of num. "zeros"

← roots of denom. "poles"

zeros increase mag & phase

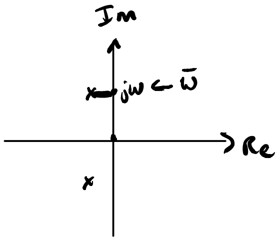
poles decrease mag & phase

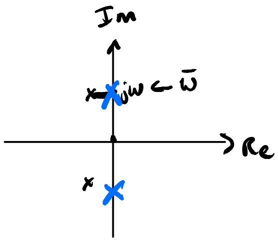


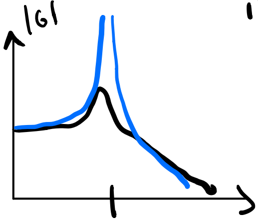
phase : smaller for larger ω

zeros : increase gain

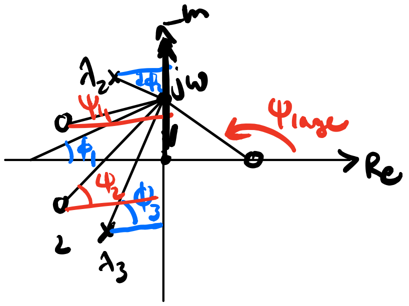
poles : decrease gain







if λ on jω axis
purely oscillating mode

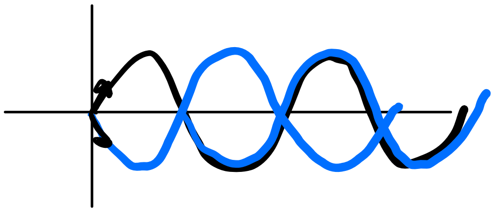


G(s) gets multiplied by $e^{i\psi_{large}}$

$$\text{phase} = \Sigma\psi - \Sigma\phi + \psi_{large}$$

when $\omega = 0$, $\psi_{large} = -180$

all other phases cancel out.



RHP zeros problems for performance

So both RHP zeros & time delays cause phase shift problems

Time delays

f(t)

↓ time delay

$$\underbrace{u(t-\tau)}_{\text{step function}} f(t-\tau) \xrightarrow{\mathcal{L}(\cdot)} \underbrace{e^{-j\tau}}_{\text{phase shift from time delay}} \mathcal{L}(f(t)$$

State Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \implies y(s) = \underbrace{C(sI-A)^{-1}x_0}_{\text{transient}} + C(sI-A)^{-1}Bu + Du$$

$$G(s) = C(sI-A)^{-1}B + D$$

diagonalize $A = P\underset{\sim}{E}P^{-1}$

$\mathrm{diag}(\lambda_1 \cdots \lambda_n)$

$$= \underset{\bar{C}}{\underset{\sim}{CP}}(sI-E)^{-1}\underset{\bar{B}}{\underset{\sim}{P^{-1}B}} + D$$

$$= \bar{C}\begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix}\bar{B} + D$$

SISO:

single input single output

$C \in \mathbb{R}^{1\times n}$, $B \in \mathbb{R}^{n\times 1}$, $D \in \mathbb{R}^{1\times 1}$

$$G(s) = \underbrace{\frac{\bar{C}_1\bar{B}_1}{s-\lambda_1} + \cdots + \frac{\bar{C}_n\bar{B}_n}{s-\lambda_n}}_{\text{superposition}} + \underset{\text{constant gain}}{\underset{\sim}{D}}$$

polynomial form

$$(s-\lambda_1)\cdots(s-\lambda_n) = \det(sI-A)$$

$$G(s) = \frac{\bar{C}_1\bar{B}_1\prod_{k\neq 1}(s-\lambda_k) + \cdots + \bar{C}_n\bar{B}_n\prod_{k\neq n}(s-\lambda_n) + D\det(sI-A)}{\det(sI-A)}$$

$\det(sI-A) \rightarrow$ eigenvalues of A are the poles of the transfer function

MIMO:

multi input multi output

$C \in \mathbb{R}^{o\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{o\times m}$

$$G(s) = C(sI-A)^{-1}B + D$$

Matrix of transfer functions

$$\begin{bmatrix} y_1(s) \\ \vdots \\ y_o(s) \end{bmatrix} = \begin{array}{c|c} & \ell \downarrow \\ \hline k \rightarrow & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} \end{array} \begin{bmatrix} u_1(s) \\ \vdots \\ u_m(s) \end{bmatrix}$$

$G_{k\ell}(s)$: tells me how $u_\ell(s)$ affects $y_k(s)$

$$G(s) = \begin{bmatrix} — c_1^T — \\ \vdots \\ — c_0^T — \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B_1 \cdots B_n \end{bmatrix} + D .$$

$$G_{kl}(s) = c_k^T (sI - A)^{-1} B_l + D_{kl}$$ ← scalar eqn

---

$$G(s) = \underbrace{\underbrace{CP}_{\bar{C}}}_{\text{rows}} (sI - E)^{-1} \underbrace{\underbrace{P^{-1}B}_{\bar{B}}}_{\text{cols}} + D$$

$$= \begin{bmatrix} q_1 \cdots q_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} — h_1^T — \\ \vdots \\ — h_n^T — \end{bmatrix} + D$$

$$= \sum_k \frac{1}{s-\lambda_k} \underbrace{q_k h_k^T}_{\text{rank 1}} + D$$

output mode direction ← q_k

h_k^T → input mode direct

$$(sI - A)^{-1} = \frac{1}{\det(sI-A)} \underline{\mathrm{Adj}(sI-A)^{-1}}$$

$$G(s) = \frac{1}{\det(sI-A)} \left(\sum_k \underline{\prod_{l \neq k} (s - \lambda_l)} q_k h_k^T + D \det(sI-A) \right)$$

$$\updownarrow$$

Polynomial...

poly order n-1 ↙ poly order n ↙

$$G(s) = \frac{1}{\underbrace{\det(sI-A)}_{\text{poles}}} \left[\underbrace{C \, \mathrm{Adj}(sI-A) B + D \underline{\det(sI-A)}} \right]$$

poly of order n ↗

poles

all dynamic information

zeros are values of s where this matrix drops rank...

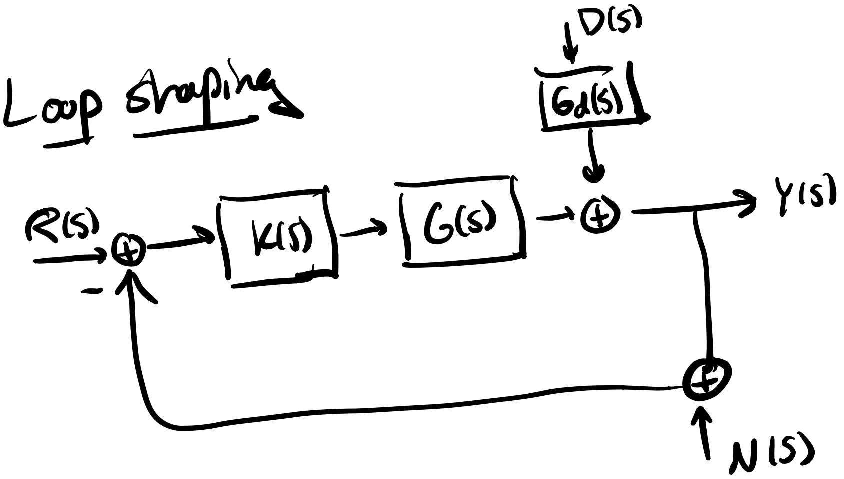
comes along w/ a zero input & output direction

Transfer Function to state-space

Not unique
- diff coord. representations,

Careful to not have uncontroll. unobs. modes $\Rightarrow$ minimal realization of transfer function

Loop shaping



$$Y(s) = \underbrace{(I+GK)^{-1}}_{T}GKR + \underbrace{(I+GK)}_{S}G_dD - \underbrace{(I+GK)^{-1}}_{T}GKN$$

L: GK   Loop

S: $(I+L)^{-1}$ sensitivity

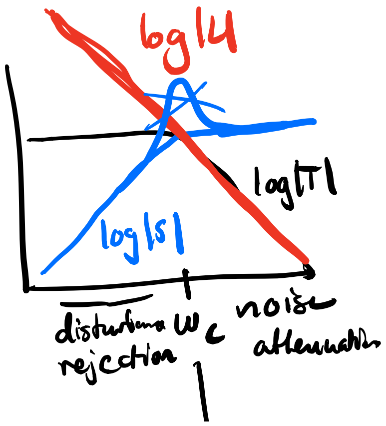
$\underline{S+T=I}$

T: $(I+L)^{-1}L$ comp sensitivity

$= L(I+L)^{-1}$

$$\underbrace{E}_{\text{want to be } 0} = Y-R = SR - \underbrace{SD}_{\text{want to be small at low freq.}} + \underbrace{TN}_{\text{want to be small at high}}$$

to look like $\frac{1}{s}$



How do S & T relate to L?   cro   ss over freq.

$S = (I+L)^{-1}$: want L large for small ω

$T = (I+L)^{-1}L$: want L small at high ω

$I - S$

ROBUSTNESS: STABILITY MARGINS

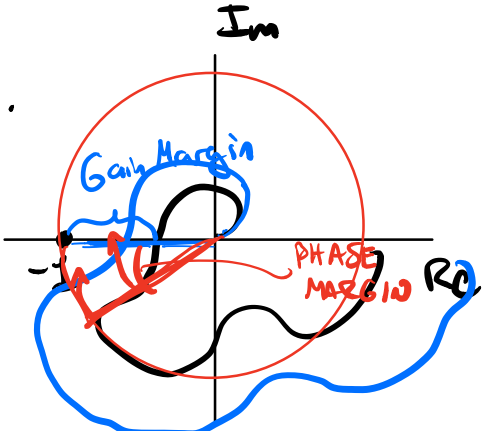
How we compensate for uncertainty in system

- extra gain & extra phase

How to characterize...

Nyquist Plot: another way to visualize a tf.

transfer function $L(j\omega)$ → complex # in scalar case



if multiply $L(j\omega)$ by a gain...

if multiply $L(j\omega)$ by a phase

Nyquist Stability Criterion:

stability of closed related to # of times $L(j\omega)$ encircles -1

$S(j\omega)$: $L(j\omega)$ goes through -1.

then $S(j\omega)$ blows up.

SISO: $S = (1+L)^{-1}$ $L = -1$ $S \to \infty$

MIMO: don't want eigenvalues of L to pass through -1 think about diagonalizing L

$S = (I+L)^{-1}$

MIMO Nyquist criteria:

similar to SISO ... uses the $\det(L(s))$

Goal is to keep $L(s)$ away from -1

DO THIS BY TRYING TO MINIMIZE $\underbrace{\max_{\omega} |S(j\omega)|}_{\text{pressing down the peak of S}}$

SISO: just consider worst case frequency

MIMO: consider
- worst case frequency
- worst case input direction

$$\min \max_{\omega, |d|=1} \Big|\underbrace{S(j\omega)}_{\text{matrix}}\underbrace{d}_{\text{vector}}\Big|_2^2 = \max_{\omega, d\neq 0} \frac{|S(j\omega)d|_2^2}{|d|_2}$$

$$\max_{\omega, d} \frac{d^* S^* S d}{d^* d} = \max_{\omega} \underbrace{\bar{\sigma}\big(S(j\omega)\big)}_{\text{maximum singular value at } j\omega}$$

Singular Value Decomposition:

Any $M \in \mathbb{C}^{m\times n}$ can be written $M = \underset{\text{cols: left singular vectors}}{U}\ \Sigma\ \overset{\text{rows are right singular vectors}}{V^*}$ $\qquad \underset{\text{singular values}}{\Sigma = \begin{bmatrix}\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k\end{bmatrix}}$

$U \in \mathbb{C}^{m\times m}$

$V \in \mathbb{C}^{n\times n}$

$\underbrace{\phantom{V \in \mathbb{C}^{n\times n}}}_{\text{unitary}}$

$U^* U = I$

$V^* V = I$

$$= \begin{bmatrix} \underset{\substack{\text{orthonorm.}\\ \text{basis}\\ R(M)}}{U_1} & \underset{\substack{\text{orth norm}\\ \text{basis}\\ N(M^*)}}{U_2}\end{bmatrix} \begin{bmatrix}\Sigma & 0 \\ 0 & 0\end{bmatrix} \begin{bmatrix} -V_1^*- \\ -V_2^*- \end{bmatrix}$$

V_1^*: o.n. basis $R(M^*)$, V_2^*: $N(M)$

$R(M) \overset{\perp}{\oplus} N(M^*) \qquad R(M^*) \overset{\perp}{\oplus} N(M)$

$$M = U \underline{\Sigma} \underline{\underline{V^*}}$$

another & rotation ③ pos. stretching ② rot. ①

$$\Sigma = \begin{bmatrix} \bar{\sigma} & \cdots & 0 \\ & \underline{\sigma} & \\ 0 & & 0 \end{bmatrix}$$

max sing. value

min sing value

worth thoroughly understanding

wikipedia

$$S = U \Sigma V^* \qquad |Sd|^2 = d^* S^* S d$$
$$= \underline{d^* V} \Sigma^2 \underline{V^* d}$$

right max sing vector

⇒ max gain direction

$$\min \left(\max_{\omega} \bar{\sigma}\big(S(j\omega)\big) = |S(j\omega)|_{\infty} \right)$$

H_∞ norm : max gain in the max gain direction over any frequency

"pushing down the peak of the transfer function"

time dom interpretation : game between controller & disturbance

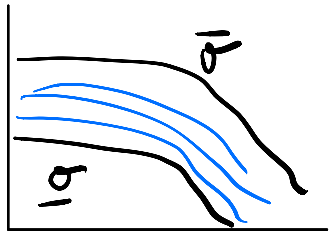
this type of design is called H_∞ design

Other uses for SVD and transfer functions

BODE PLOTS (MIMO)

max and min σ

ex 1.

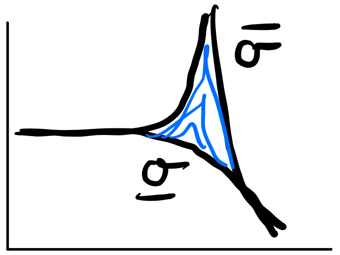


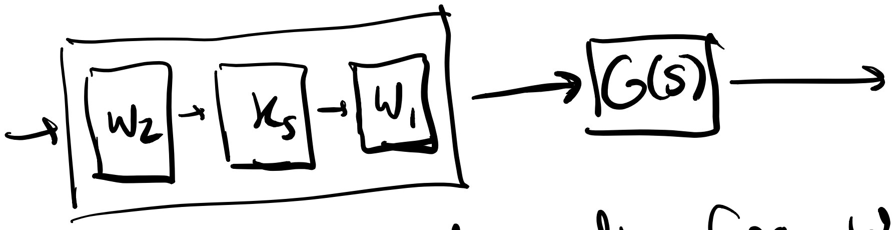
Pre and Post - Compensator

control design

decoupling MIMO systems to design SISO controllers

ex 2.





- pick a desired operating freq. ω_0 ← often around cross over freq.
- compute SVD $G(j\omega_0)$

$$G(j\omega_0) = U_0 \Sigma_0 V_0^*$$

const gains

choose $\underline{W_1} = V_0$, $\underline{W_2} = U_0^*$

select $K_s = \underline{l(s)}\,\Sigma_0^{-1}$

like integrator

$$K_s = \begin{bmatrix} l_1(s) & & \\ & \ddots & \\ & & l_n(s) \end{bmatrix} \Sigma_0^{-1}$$

where these are similar

const gain

$$L(j\omega_0) = G(j\omega)K(j\omega)$$

$$= G(j\omega)W_1 K_S W_2$$

$$= U_0 \Sigma_0 V_0^* V_0 \begin{bmatrix} l_1(s) & & \\ & \ddots & \\ & & l_k(s) \end{bmatrix} U_0^*$$

$$= \underbrace{U_0}_{\downarrow} \begin{pmatrix} l_1(s) & & \\ & \ddots & \\ & & l_k(s) \end{pmatrix} U_0^*$$

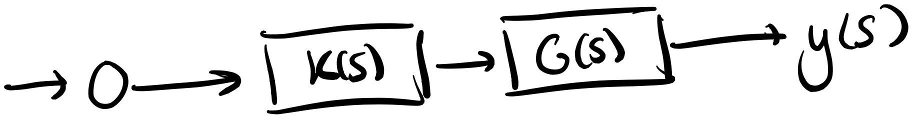
orthonormal input output directions

→ SISO loops (integrators w/ desired cross freq, etc.)

decoupling system at ω_0

NEXT CLASS:

H_∞, H_2



Before: constant feedback

but K is function of s... not constant

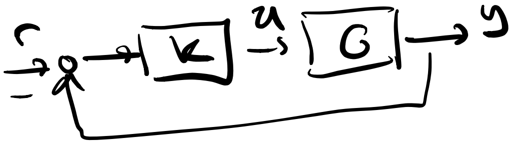
has some states...

Plant

$$\dot{x} = Ax + Bu + B_d d + w$$
$$y = Cx + Du + v$$

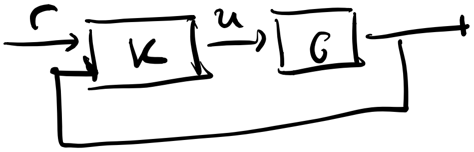
Controller

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}(r-y)$$
$$u = \tilde{C}\tilde{x} + \tilde{D}(r-y)$$



1 DOF

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\begin{bmatrix} r \\ y \end{bmatrix}$$



2 DOF

$$u = \tilde{C}\tilde{x} + \tilde{D}\begin{bmatrix} r \\ y \end{bmatrix}$$

Examples

① Full state feedback $r=0, \; y=x$ → $u = Kx$

nor dynamics

$\tilde{A}=0, \; \tilde{B}=0, \; \tilde{C}=0, \; \tilde{D}=K$

② LQG control

want to drive x to r

$$u = K(\hat{x}-r) = K\hat{x} - Kr$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x}-y)$$
$$\underbrace{\dot{\hat{x}}}_{\tilde{x}} = \underbrace{[A+BK+LC]}_{\tilde{A}}\hat{x} + \underbrace{[-BK \;\; -LC]}_{\tilde{B}}\begin{bmatrix} r \\ y \end{bmatrix}$$
$$u = \underbrace{K}_{\tilde{C}}\hat{x} + \underbrace{[-K \;\; 0]}_{\tilde{D}}\begin{bmatrix} r \\ y \end{bmatrix}$$

write down LQG controller
↓
take transfer function
and apply