Questions:

Hamiltonian systems: $\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \nabla f$ $\quad f(x, \lambda)$

feasibility

$$\dot{x} = -\frac{\partial f}{\partial x} \qquad \dot{\lambda} = \frac{\partial f}{\partial x} \Rightarrow \dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

optimality

q: gen coords $\quad$ H: energy

p: momentum

$Ax + Bu$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \Rightarrow \dot{x} = Ax - \underbrace{BR^{-1}B^T\lambda}_{u = R^{-1}B^T\lambda} \qquad \lambda = Px$$

$\dot{x}$

$\dot{\lambda} = -Qx - A^T\lambda(T)$

$\lambda(T) = Q_T x(T)$

$u = R^{-1} B^T \lambda$

optimal control; R^{-1}: penalizing for control in directions you care ab-; B^T: map back to control space; λ: direction want to push state

PROBLEM 2:

a) $\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} X \\ Y \end{bmatrix}$, $\quad \boxed{\begin{matrix} X(T) = I \\ Y(T) = Q_T \end{matrix}}$

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-T)} \begin{bmatrix} X(T) \\ Y(T) \end{bmatrix} \qquad P(t) = Y(t)X(t)^{-1}$$

$e^{-H(T-t)}$

b) solve ARE.

$$A^TP + PA + Q - PBR^{-1}B^TP = 0$$

nonlinear quadratic $\rightarrow$ Hamiltonian: solve eigenvector problem ...

Lecture:

Outline:

- Adjoint Method for Optimal Control
- introduction to H_2 & H_∞ control
  high level, not on final

Adjoint Method for computing gradients for optimal control.

Trajectory planning & improvement.

BASIC NON LIN OPT:

$$\min_{u \in \mathbb{R}^n} J(u)$$

Convex, lin probs.

$$\frac{\partial J}{\partial u} = 0$$

Non lin: $\cancel{\frac{\partial J}{\partial u} = 0}$

Gradient descent

$$u^+ = u - \alpha \frac{\partial J}{\partial u}\bigg|_u$$

trying to find local optima...

add constraints...

use Lagrange multipliers.

Background

$$\min_{u \in \mathbb{R}^n} J(u)$$
s.t. $g(u) = 0$ } m equs...

Lagrangian:

$$L(u, \lambda) = J(u) + \lambda^T g(u)$$

$\lambda \in \mathbb{R}^m$

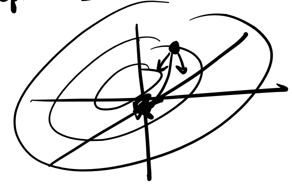
Convex, lin

$$\frac{\partial J}{\partial u} = 0 \quad \frac{\partial J}{\partial \lambda} = 0$$

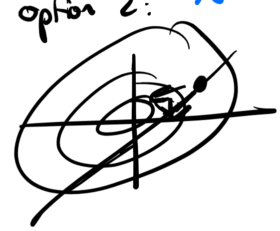
nonlinear case

projected gradient descent

option 1:



option 2: *



$\frac{\partial J}{\partial \lambda} = 0 \Rightarrow g(u) = 0$

Optimal Control: gradient descent

$\min_{u(t):\, t\in[0,T]} J(u)$

signal over time horizon

s.t. $\dot{x} = f(x,u,t)$, $x(0) = x_0$

How do we compute $\frac{\partial J}{\partial u}$ [0, T] signal

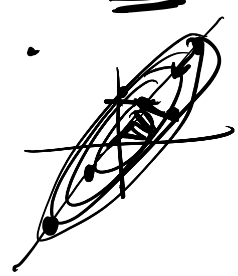
$$\boxed{\overset{+}{u}(t) = u(t) - \alpha \frac{\partial J}{\partial u}(t)}$$

How to compute: $\frac{\partial J}{\partial u}(t)$

lots of nonlin. solvers . . .

2 functions

$\underline{J(u)}$, $\underline{\frac{\partial J}{\partial u}}\left(\frac{\partial^2 J}{\partial u^2}\right) \to$ they choose α



DISCRETE TIME:

$$\min_{\substack{u[t] \\ t=0,\dots,T}} \sum_{t=0}^{T-1}\left(l[x,u,t]\right) + l[x(T),T]$$

s.t. $x[t+1] = f(x,u,t)$, $x[0] = x_0$

Trick: augment dynamics to keep track of running cost.

new state z s.t. $z[t+1] = \underline{z[t] + l[x,u,t]}$ $\quad z[0] = 0$

$$\Rightarrow z[T] = \sum_{t=0}^{T}\left(l[x,u,t]\right)$$

$z[t]$ cumulative cost up to time t

$$\min_{\substack{u[t] \\ t=0,\dots,T}} \underline{l[x[T],T] + z[T]} = \underline{\bar{J}[\bar{x}[T]]}$$

s.t. $\bar{x}[t+1] = \bar{f}[x,u,t]$ $\quad \bar{x}[0] = \bar{x}_0$

where $\bar{x}[t] = \begin{bmatrix} x[t] \\ z[t] \end{bmatrix}$

$\bar{f}[x,u,t] = \begin{bmatrix} f[x,u,t] \\ l(x,u,t) + z \end{bmatrix}$

$\bar{x}[0] = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$

$\underline{\Delta J} \simeq \boxed{\frac{\partial J}{\partial u}}\ \underline{\Delta u}$ structure

$\Delta u \rightarrow \Delta \vec{x}$

$\underbrace{\qquad\qquad}_{} \longrightarrow \Delta J$

$\Delta u \rightarrow \Delta \vec{x}$ } linearization

$$\cancelto{0}{\bar{x}[t+1]} + \Delta\bar{x}[t+1] \approx \cancelto{0}{\bar{f}} + \underbrace{\frac{\partial f}{\partial x}\bigg|_{\bar{x}(t),\bar{u}(t),t}}_{:= A_t} \Delta\bar{x} + \underbrace{\frac{\partial f}{\partial u}\bigg|_{\bar{x}(t),\bar{u}(t),t}}_{:= B_t} \Delta u$$

$$A_t = \begin{array}{c|cc} & x & \ell \\ \hline f & \frac{\partial f}{\partial x}\Big|_t & 0 \\ \ell & \frac{\partial \ell}{\partial x}\Big|_t & 1 \end{array} \qquad B_t = \begin{bmatrix} \frac{\partial f}{\partial u}\Big|_t \\ \frac{\partial \ell}{\partial u}\Big|_t \end{bmatrix}$$

$$\Delta\bar{x}[t+1] = A_t \Delta\bar{x}[t] + B_t \Delta u[t]$$ solving lin time varying.

$$\underline{\Delta\bar{x}_T} = A_{T-1}\cdots A_0 \Delta\bar{x}_0 + \left[\, A_{T-1}\cdots A_1 B_0 \,\middle|\, \cdots \,\middle|\, A_{T-1}B_{T-2} \,\middle|\, B_{T-1} \,\right] \begin{bmatrix} \Delta u_0 \\ \vdots \\ \Delta u_{T-1} \end{bmatrix}$$

$$\Delta J = \Delta\bar{J} = \frac{\partial \bar{J}}{\partial \bar{x}[T]} \Delta\bar{x}[T]$$

$$= \frac{\partial \bar{J}}{\partial \bar{x}[T]} \left[A_{T-1}\cdots A_0 \Delta\bar{x}_0 + \left[\, A_{T-1}\cdots A_1 B_0 \,\middle|\, \cdots \,\middle|\, A_{T-1}B_{T-2} \,\middle|\, B_{T-1} \,\right] \begin{bmatrix} \Delta u_0 \\ \vdots \\ \Delta u_{T-1} \end{bmatrix} \right]$$

$$\Delta J \qquad = \underbrace{\frac{\partial \bar{J}}{\partial \bar{x}[T]} \left[\, A_{T-1}\cdots A_1 B_0 \,\middle|\, \cdots \,\middle|\, A_{T-1}B_{T-2} \,\middle|\, B_{T-1} \,\right]} \begin{bmatrix} \Delta u_0 \\ \vdots \\ \Delta u_{T-1} \end{bmatrix}$$

$$= \left[\frac{\partial \ell}{\partial x[T]} \quad 1 \right] \left[\, A_{T-1}\cdots A_1 B_0 \,\middle|\, \cdots \,\middle|\, A_{T-1}B_{T-2} \,\middle|\, B_{T-1} \,\right] \begin{bmatrix} \Delta u_0 \\ \vdots \\ \Delta u \end{bmatrix}$$

Define costate $\lambda[t] =$