Graph Structures & Matrices

Algebraic Graph Theory

Acknowledgements: Mehran Mesbahi
Mathias Colbert Russelson,
Sarah Li
Shahriar Talebi

Spring 2022 - Dan Calderone

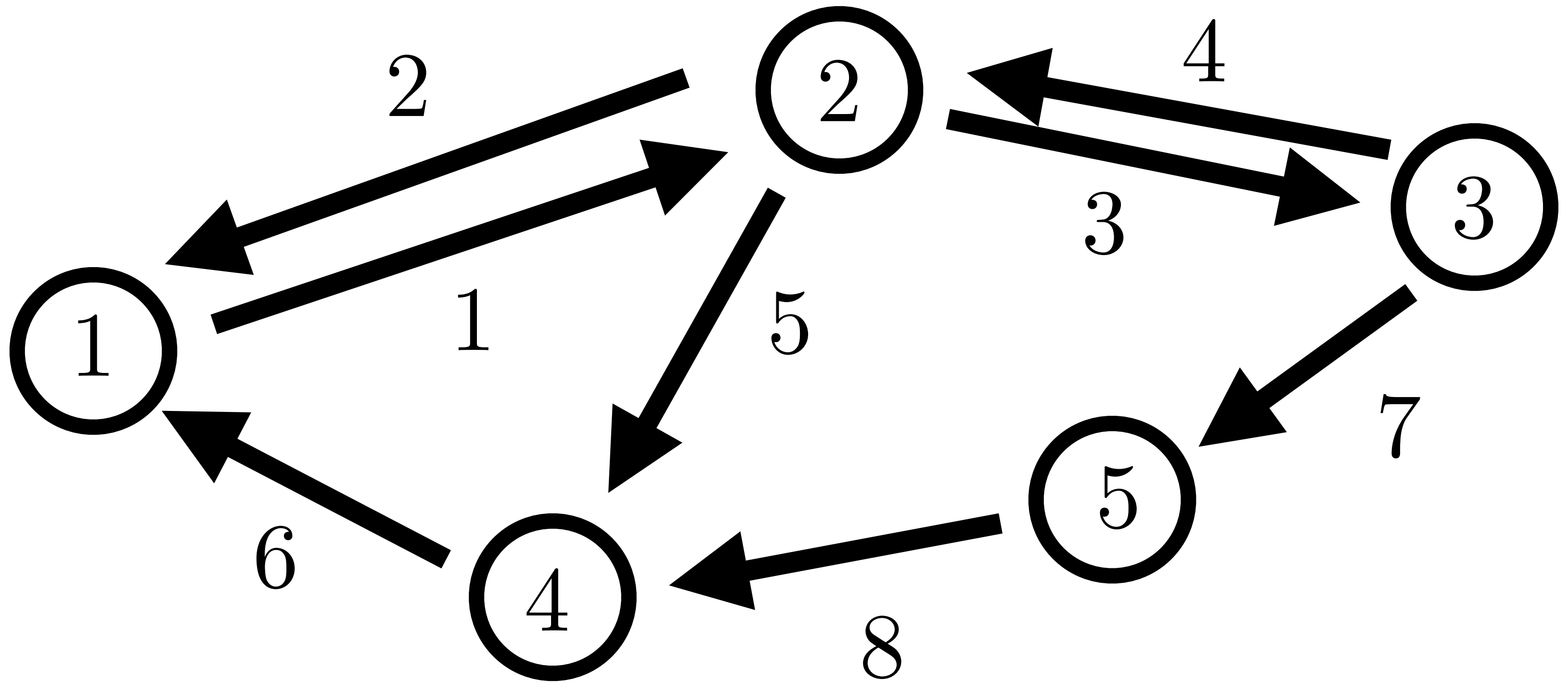
Graphs

Graph:

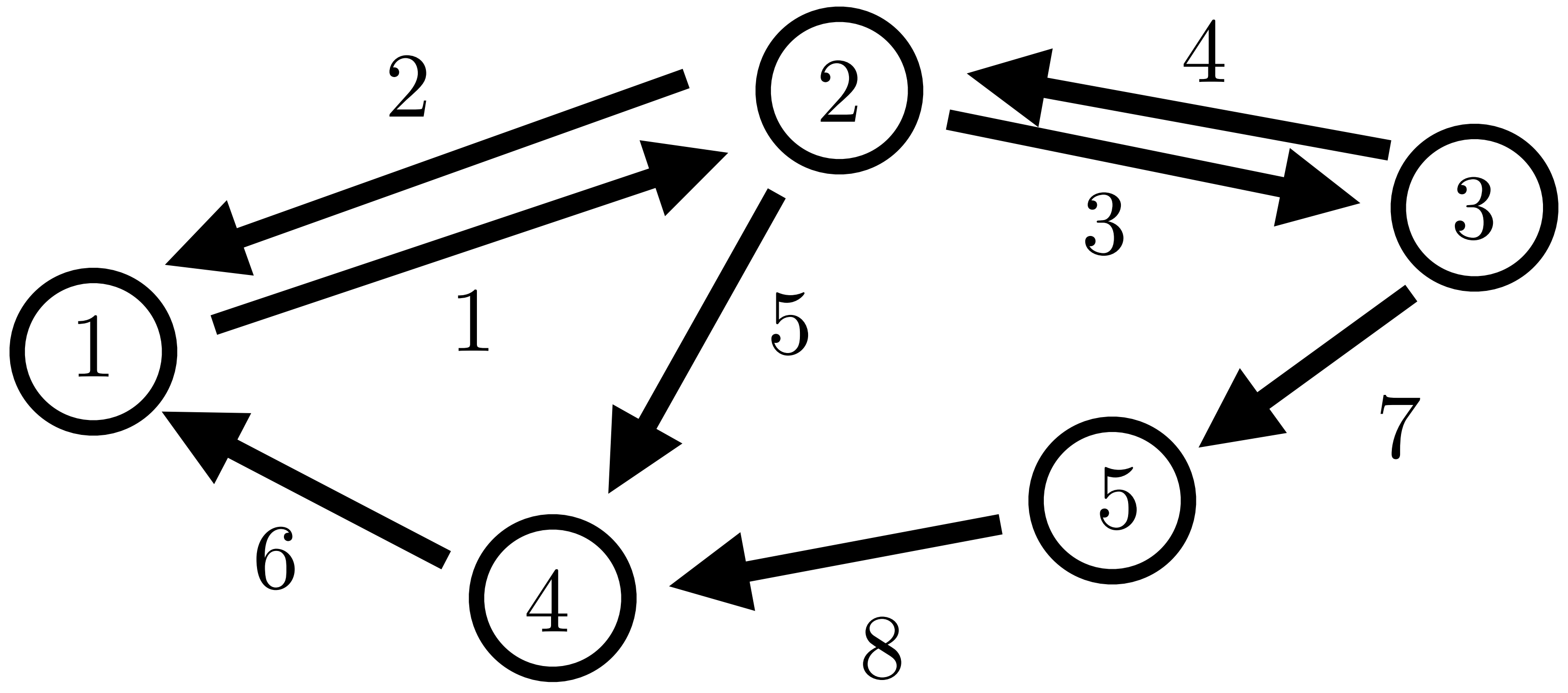
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$



Graphs



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Directed or Undirected Edges

$e = (v, v')$ edge e is "incident" to v and v'

Neighborhoods: set of "adjacent" nodes

$$\mathcal{N}_v = \left\{ v' \in \mathcal{V} \;\middle|\; e = (v, v') \in \mathcal{E} \right\}$$

(degree of vertex) $d_v = |\mathcal{N}_v|$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

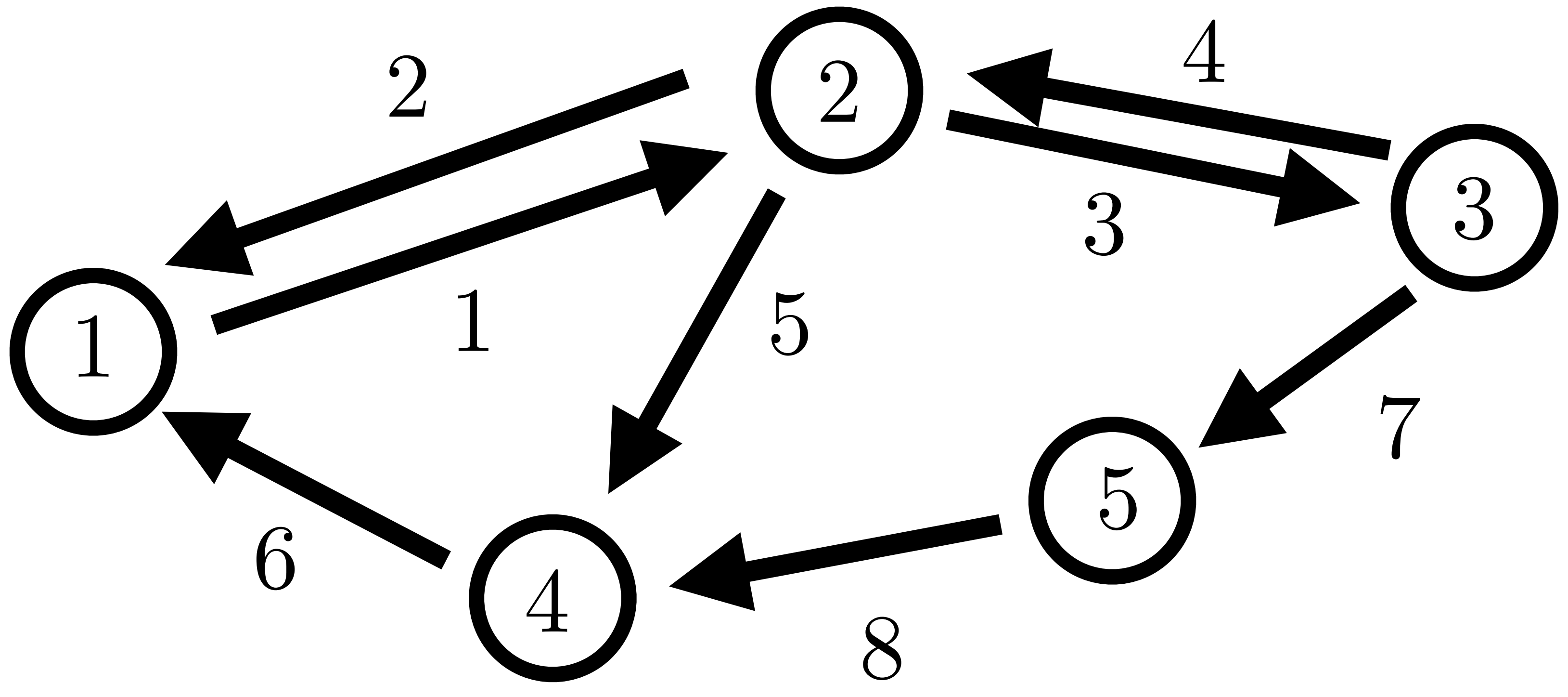
$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ out of } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

edges

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

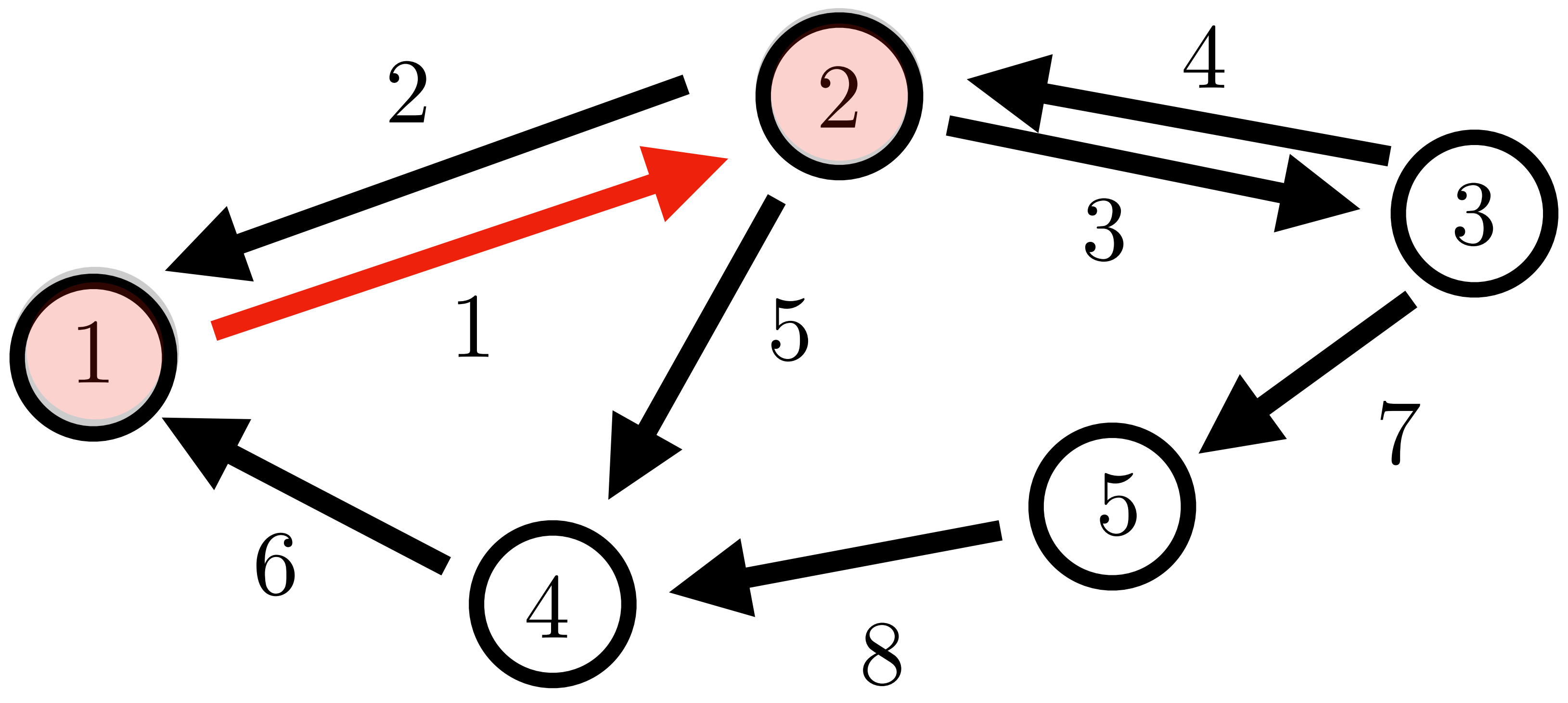
vertices



$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ out of } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$\longleftarrow$ edges $\longrightarrow$

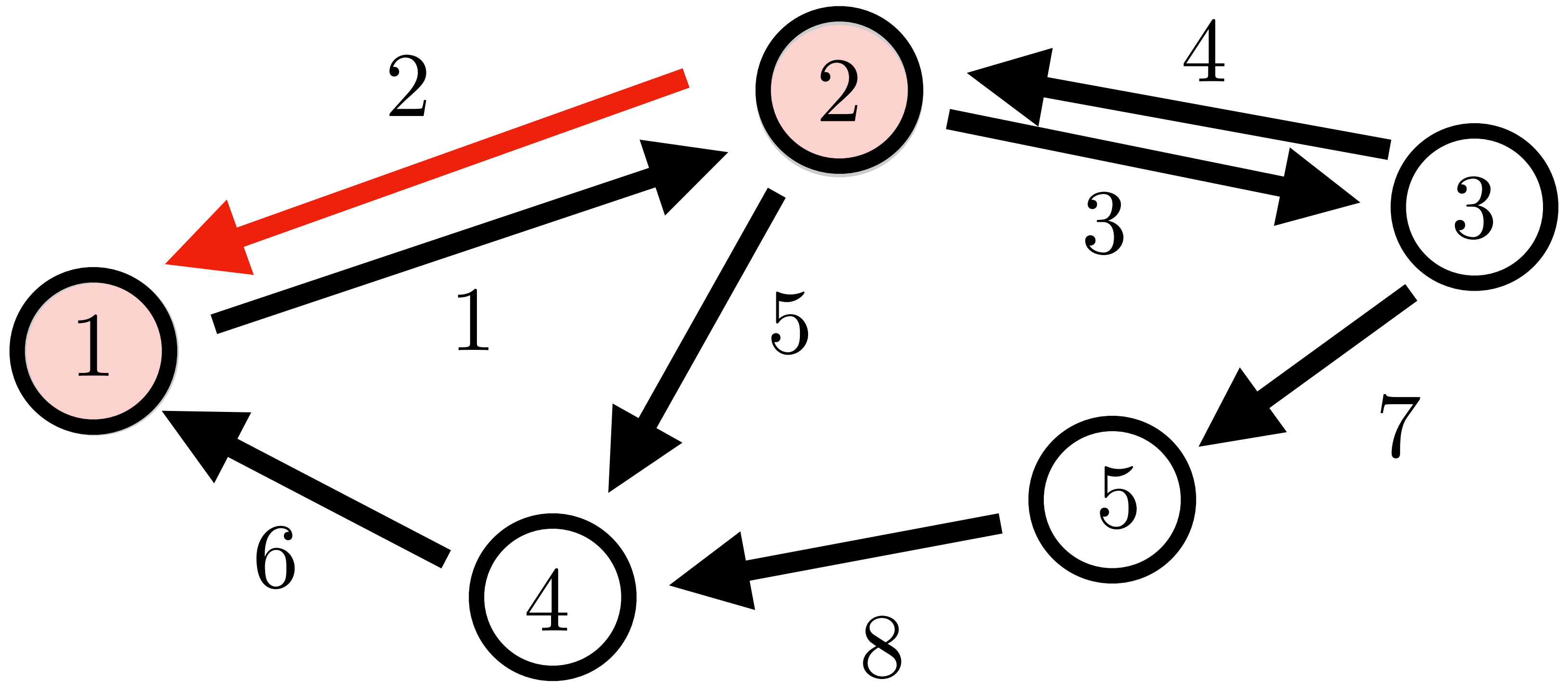
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$\uparrow$ vertices $\downarrow$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$\longleftarrow$ edges $\longrightarrow$

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$\updownarrow$ vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

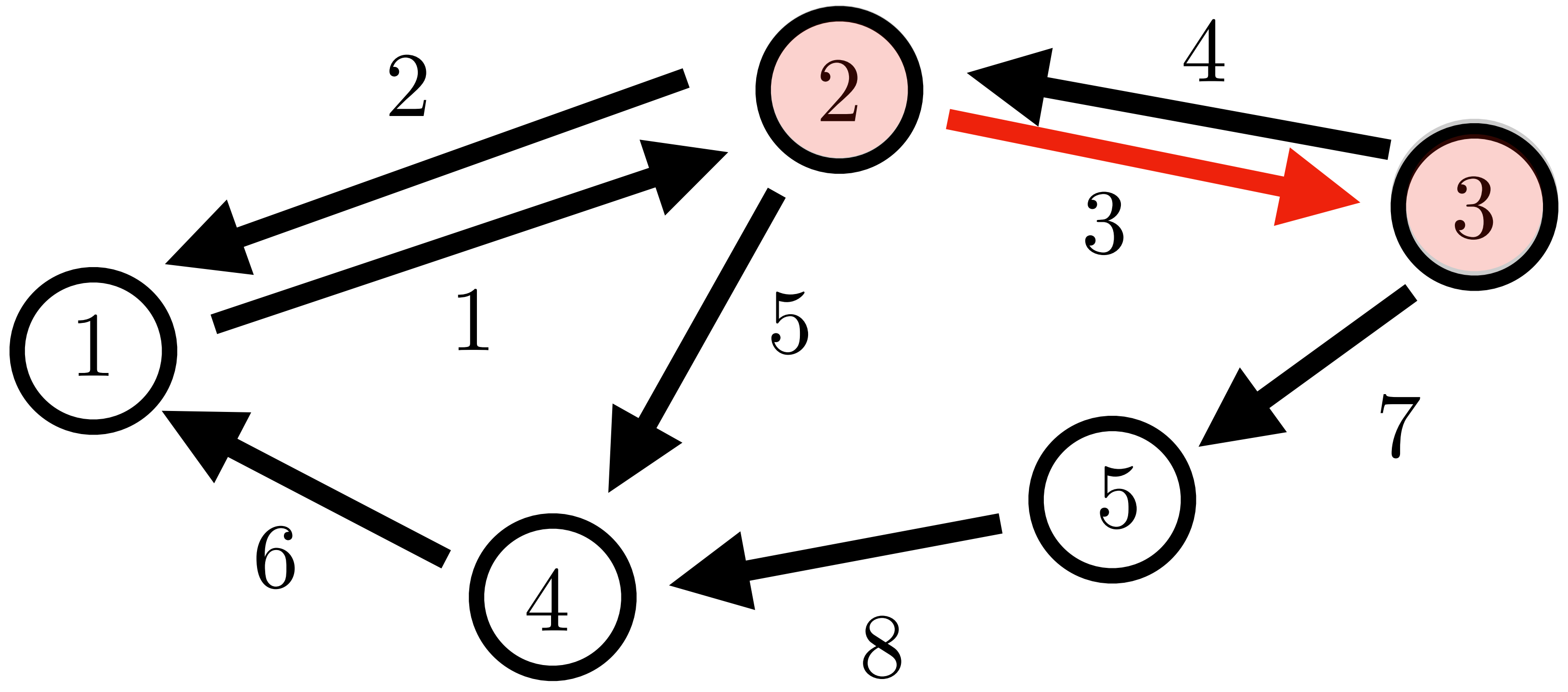
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

edges

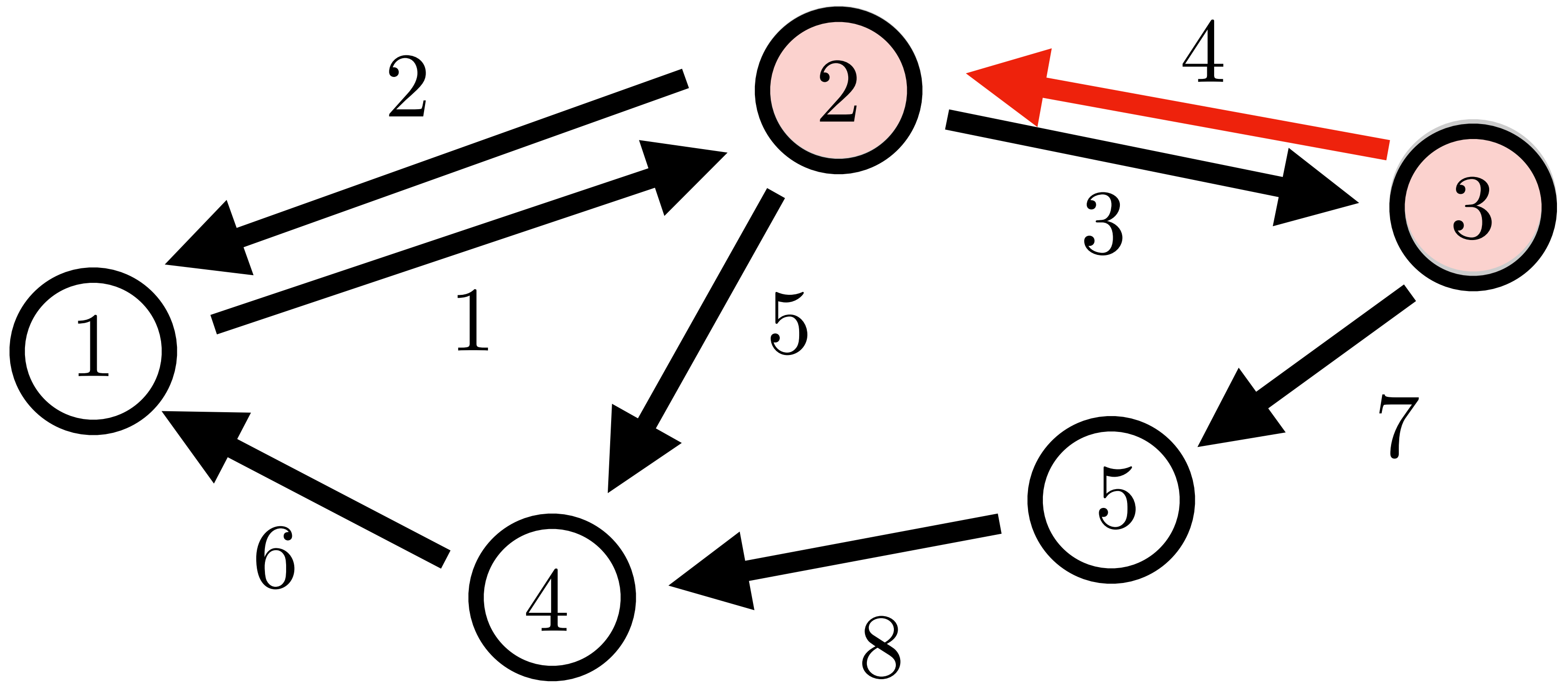
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D = \begin{array}{c} \longleftarrow \text{edges} \longrightarrow \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} \uparrow \\ \text{vertices} \\ \downarrow \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

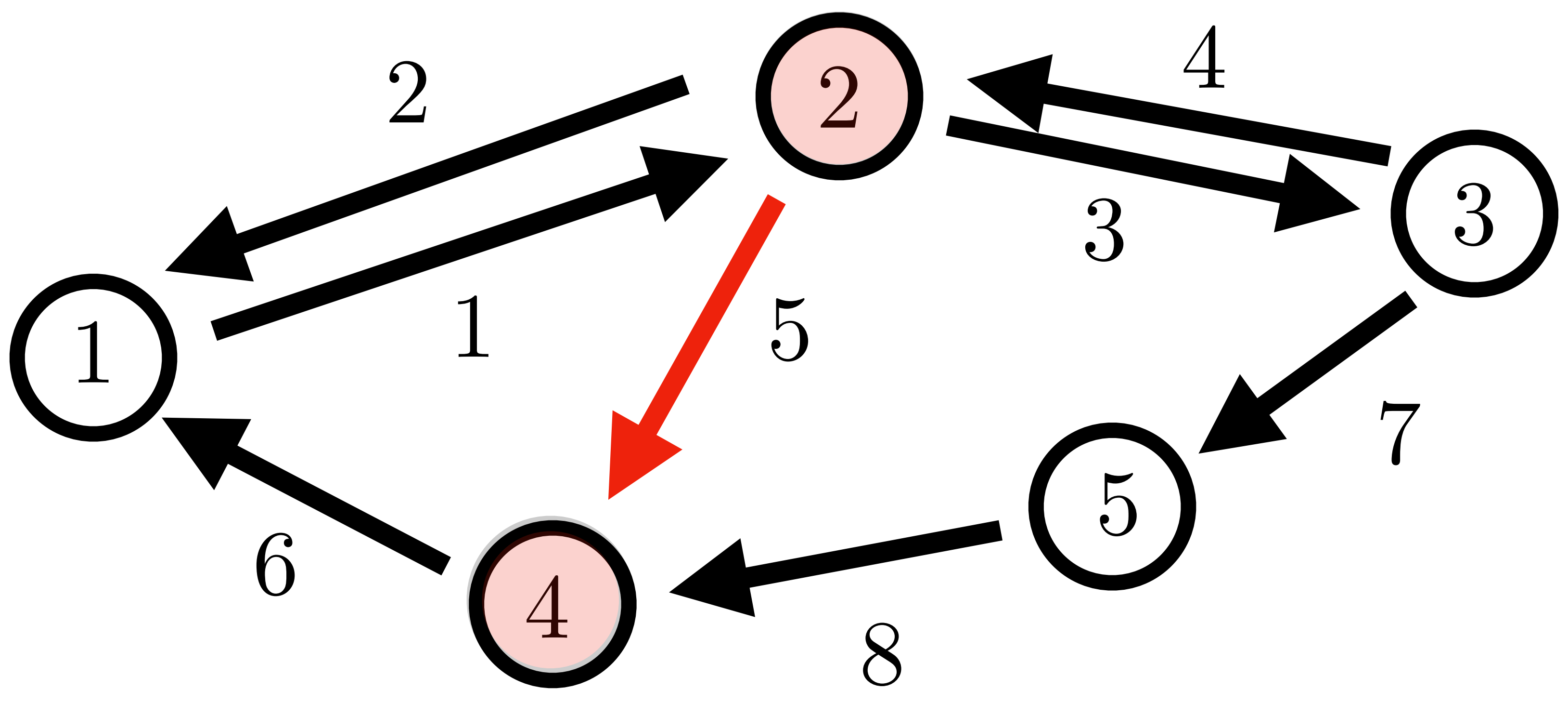
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

← edges →

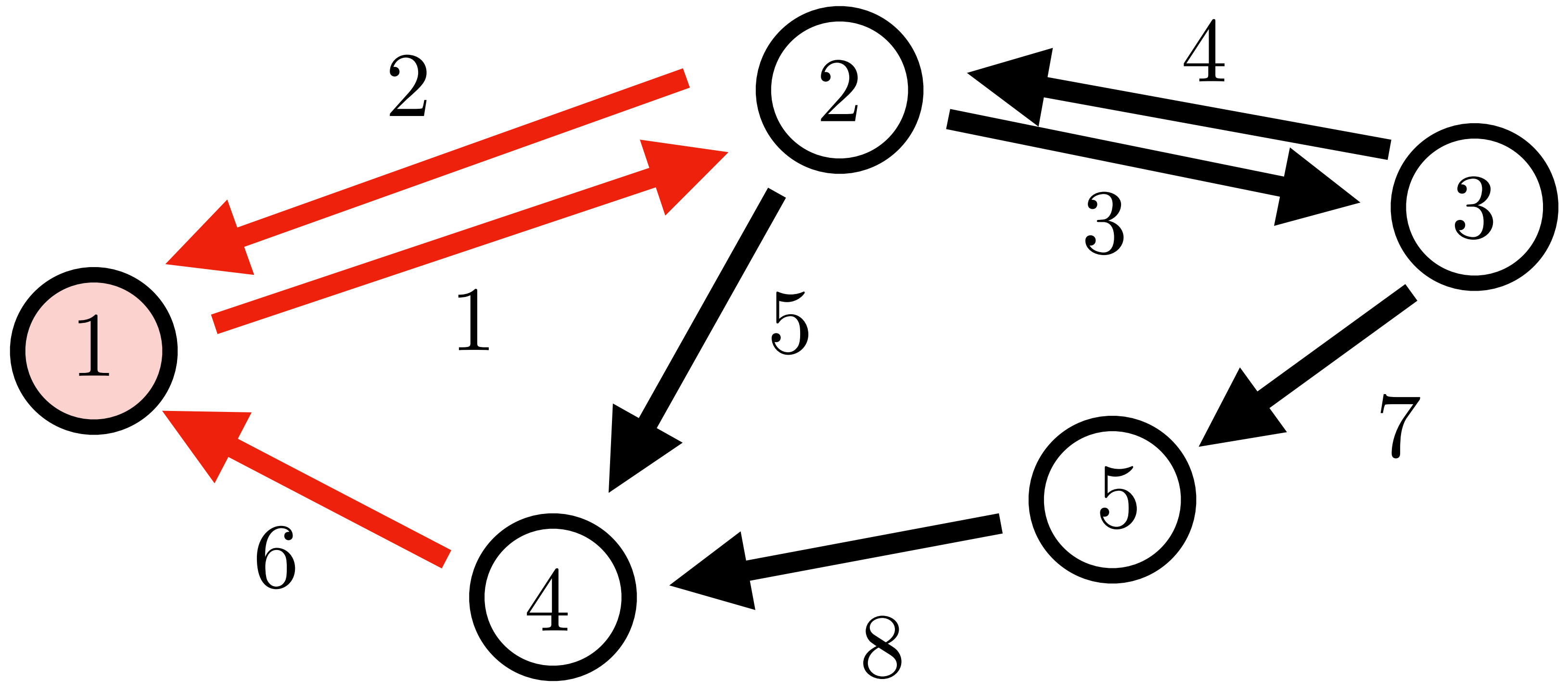
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

↑ vertices ↓

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

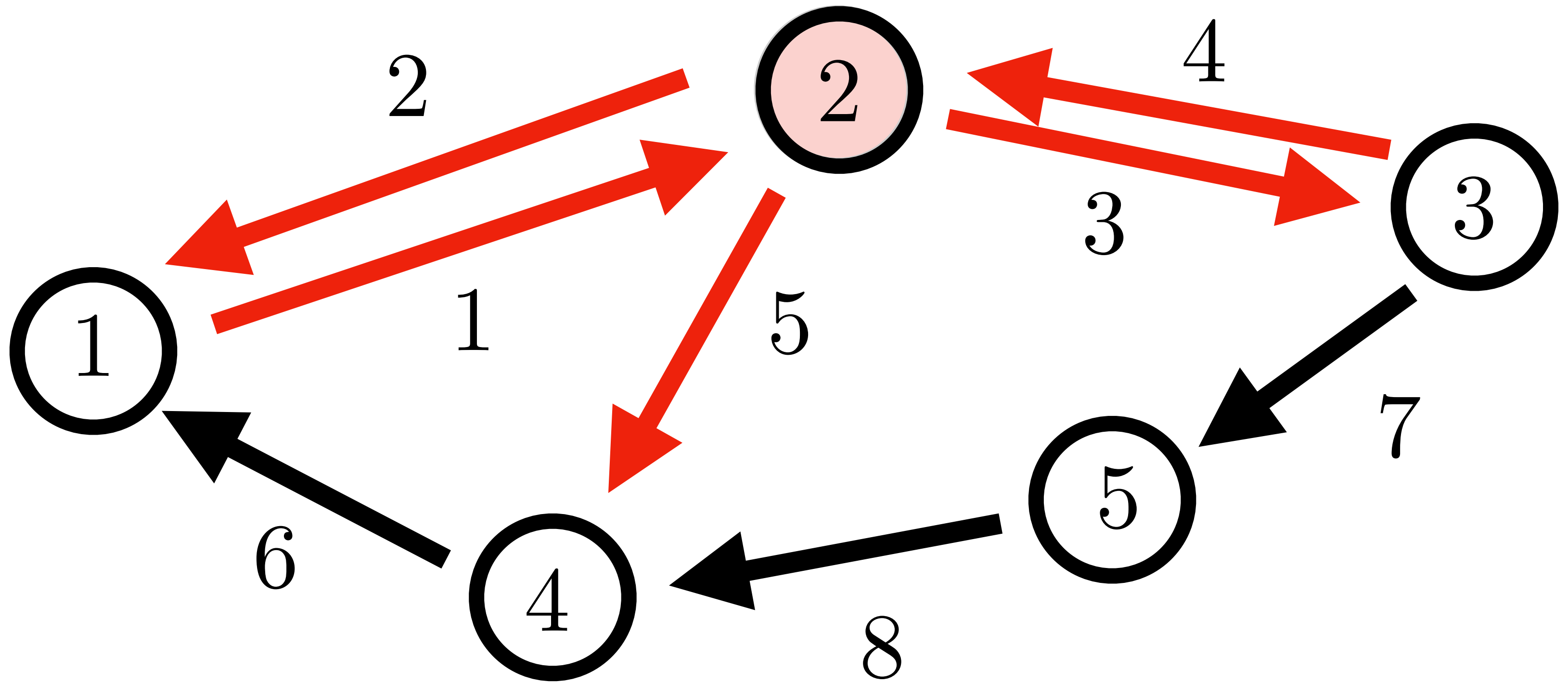
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{array}{c} \longleftarrow\ \text{edges}\ \longrightarrow \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} \uparrow \\ \text{vertices} \\ \downarrow \end{array}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

← edges →

$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

↑ vertices ↓

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

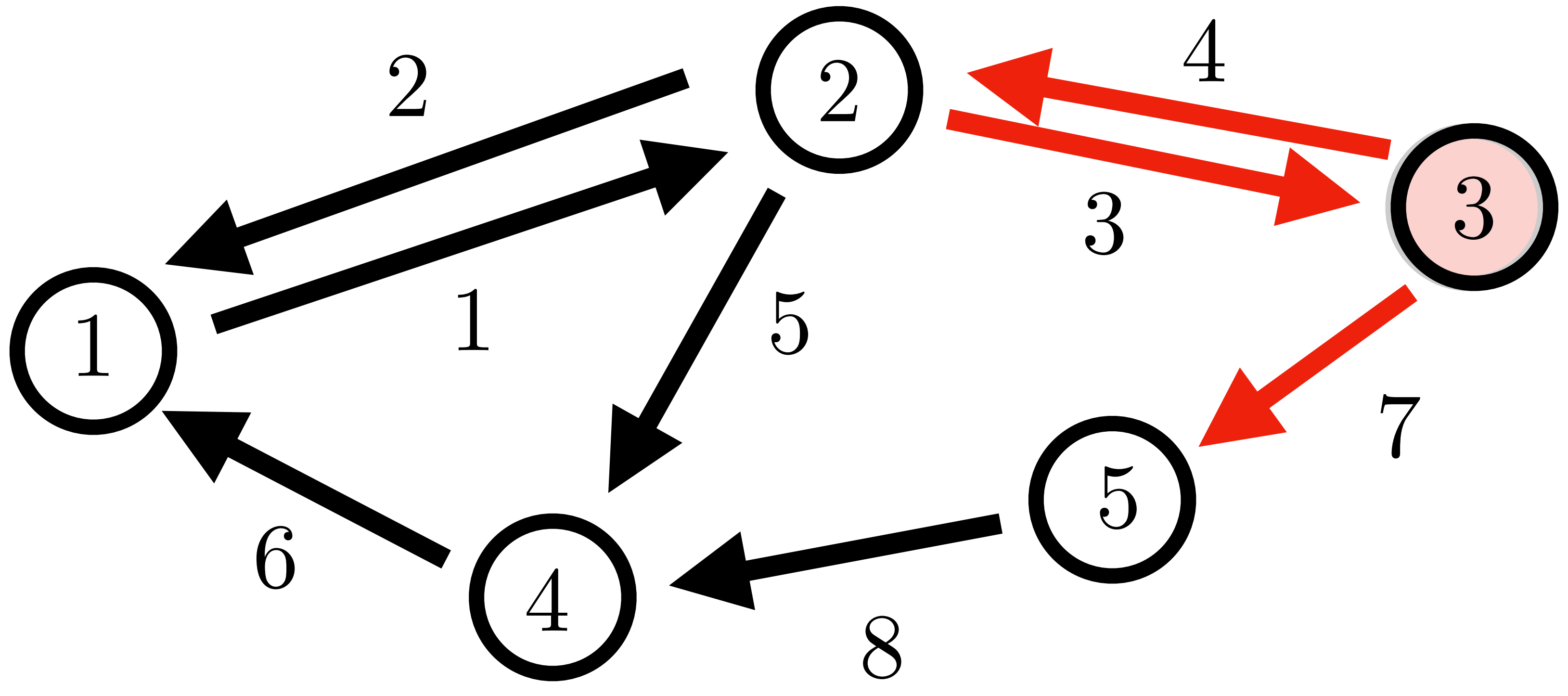
Incidence Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\qquad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$



$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

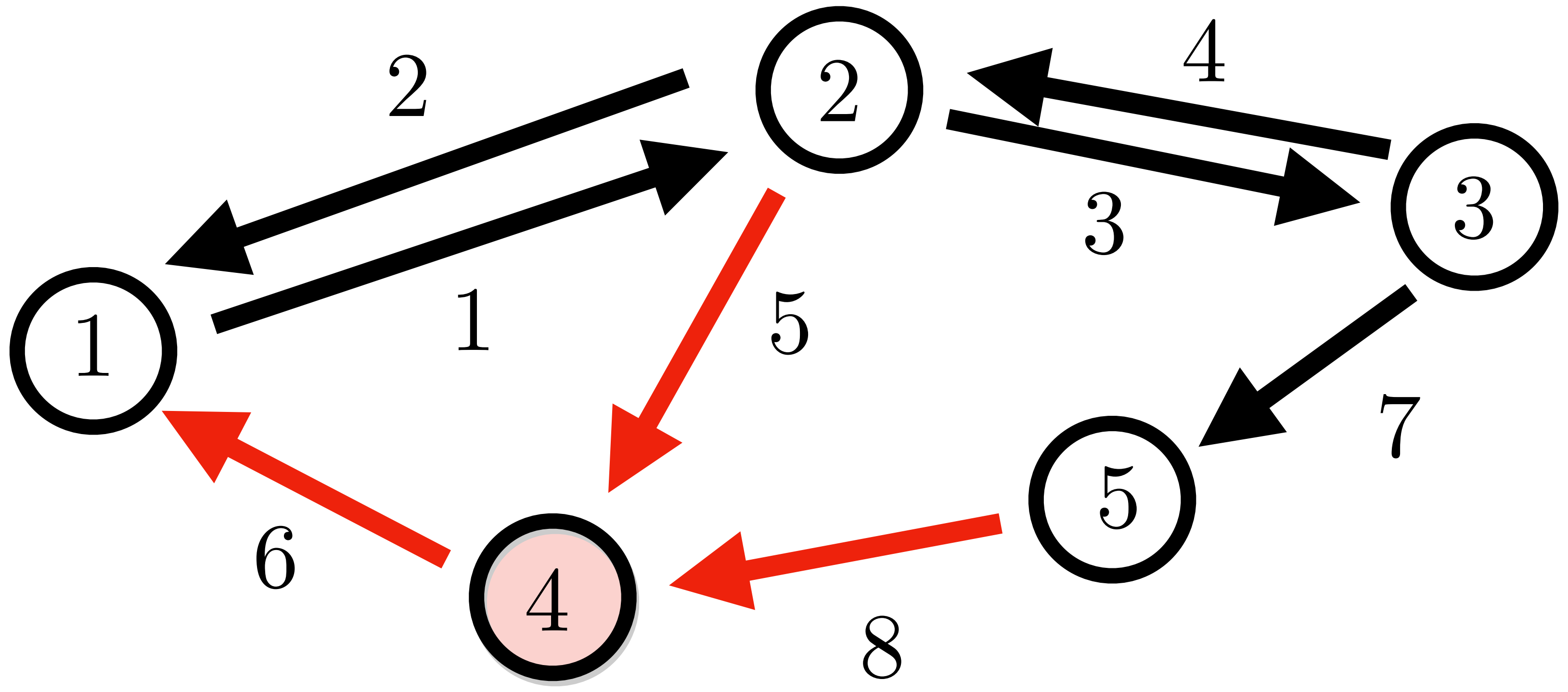
$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D = \begin{array}{cc} \longleftarrow \text{edges} \longrightarrow & \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} \uparrow \\ \text{vertices} \\ \downarrow \end{array} \end{array}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

edges

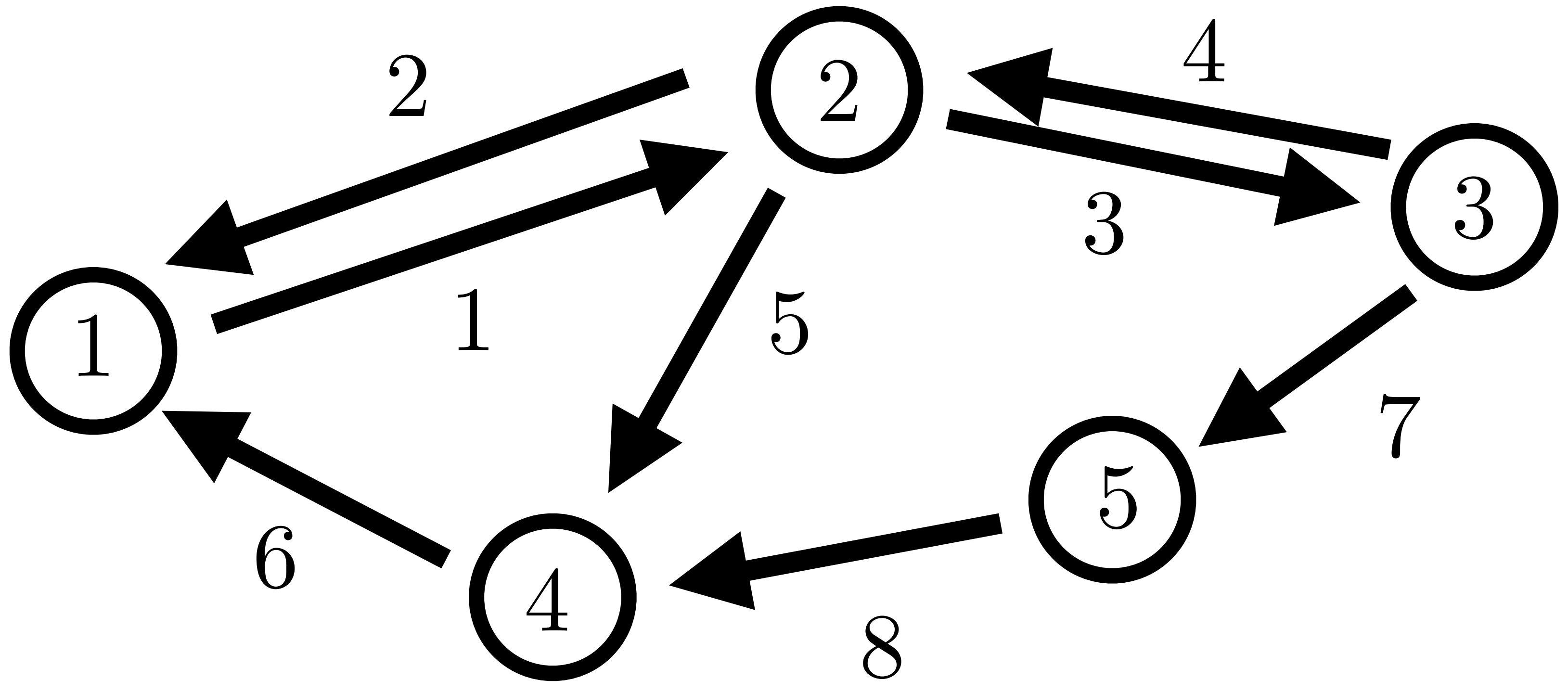
$$D = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

vertices

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix - Left Nullspace



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\ D\ \Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\ D_{\text{out}}\ \Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ out of } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\ D_{\text{in}}\ \Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \mathbf{0}$$

$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Left
Nullspace

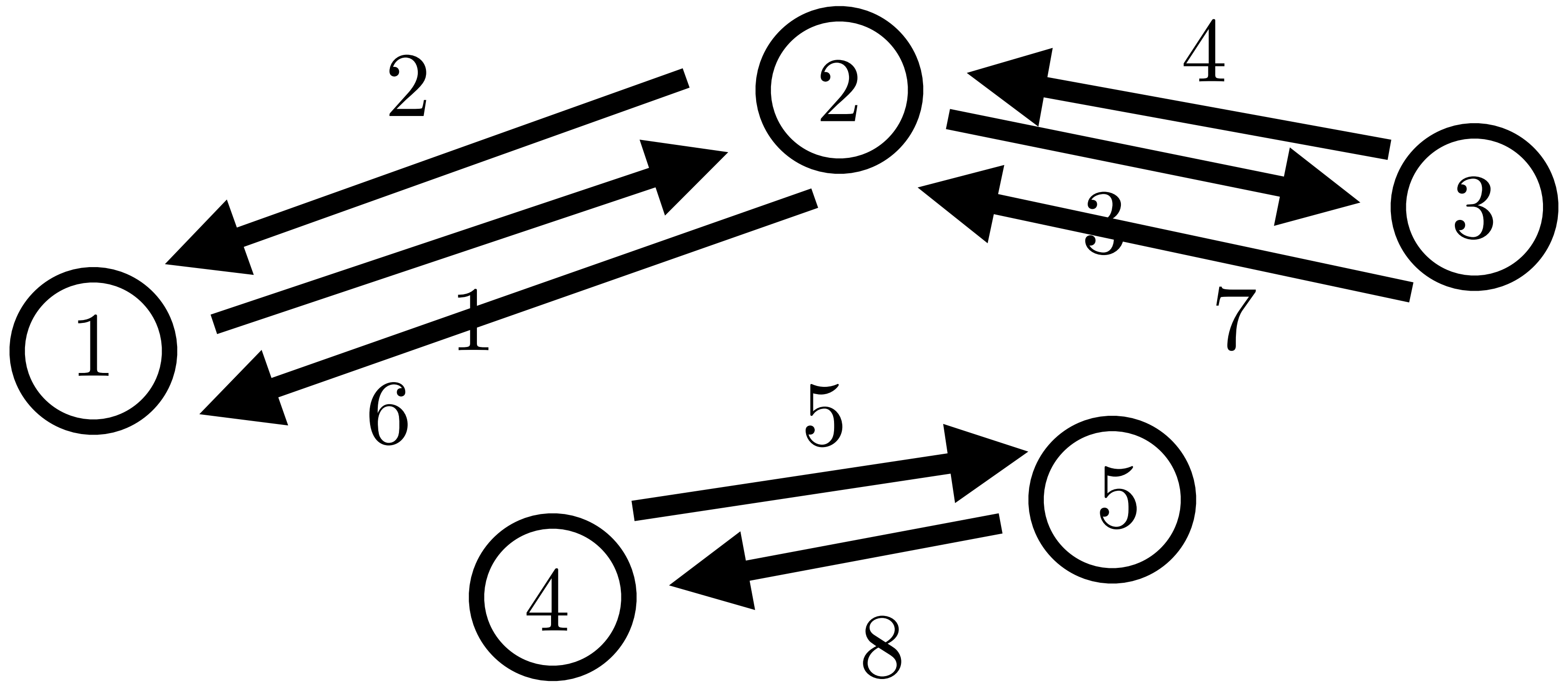
$$\mathbf{1}^T D = \mathbf{0}$$

Incidence Matrix - Left Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{out}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ out of } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\Big[\, D_{\text{in}} \,\Big]_{ve} = \begin{cases} 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

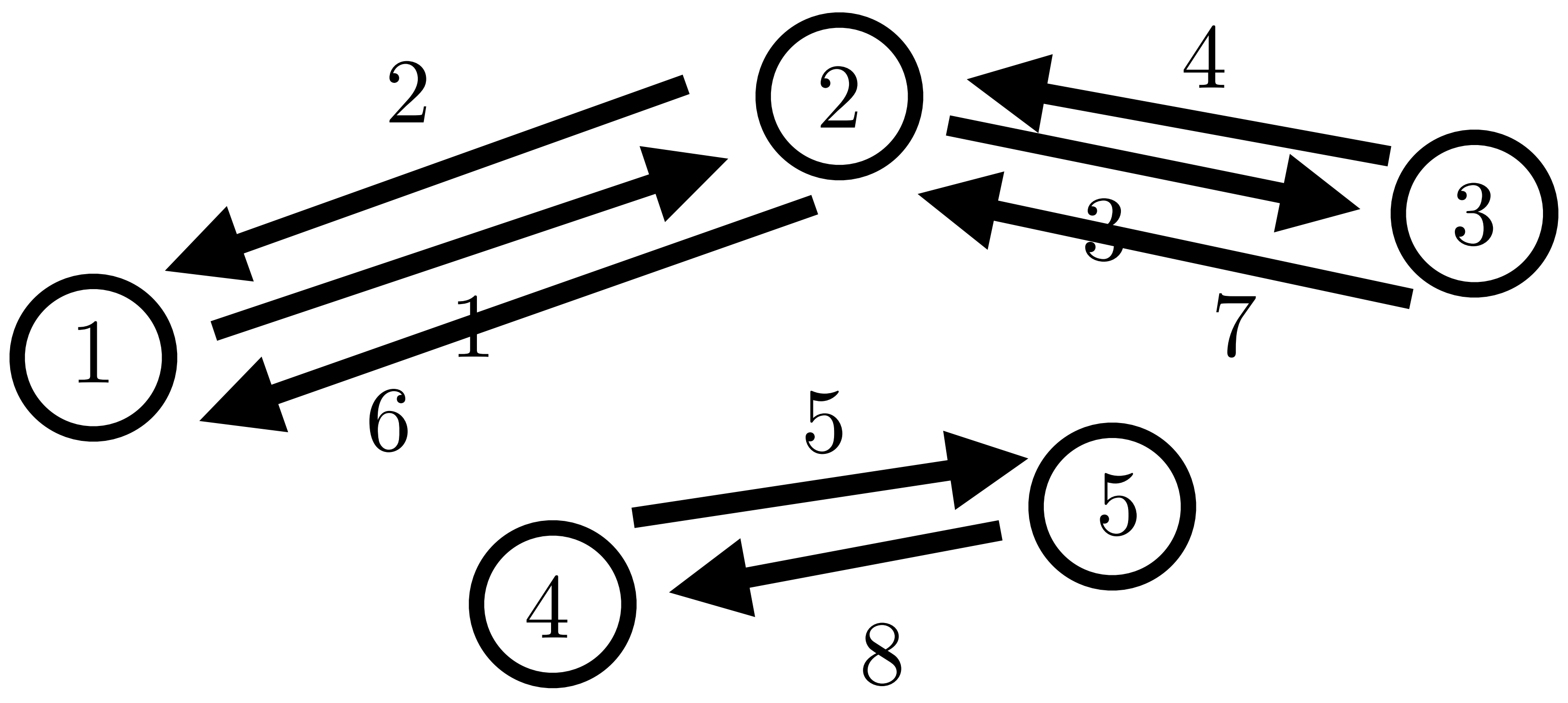
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

Left Nullspace

$$\mathbf{1}^T D = \mathbf{0}$$

Incidence Matrix - Left Nullspace



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\ D\ \Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\ D_{\text{out}}\ \Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ out of } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\Big[\ D_{\text{in}}\ \Big]_{ve} = \begin{cases} 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$D_{\text{out}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} = \mathbf{0}$$

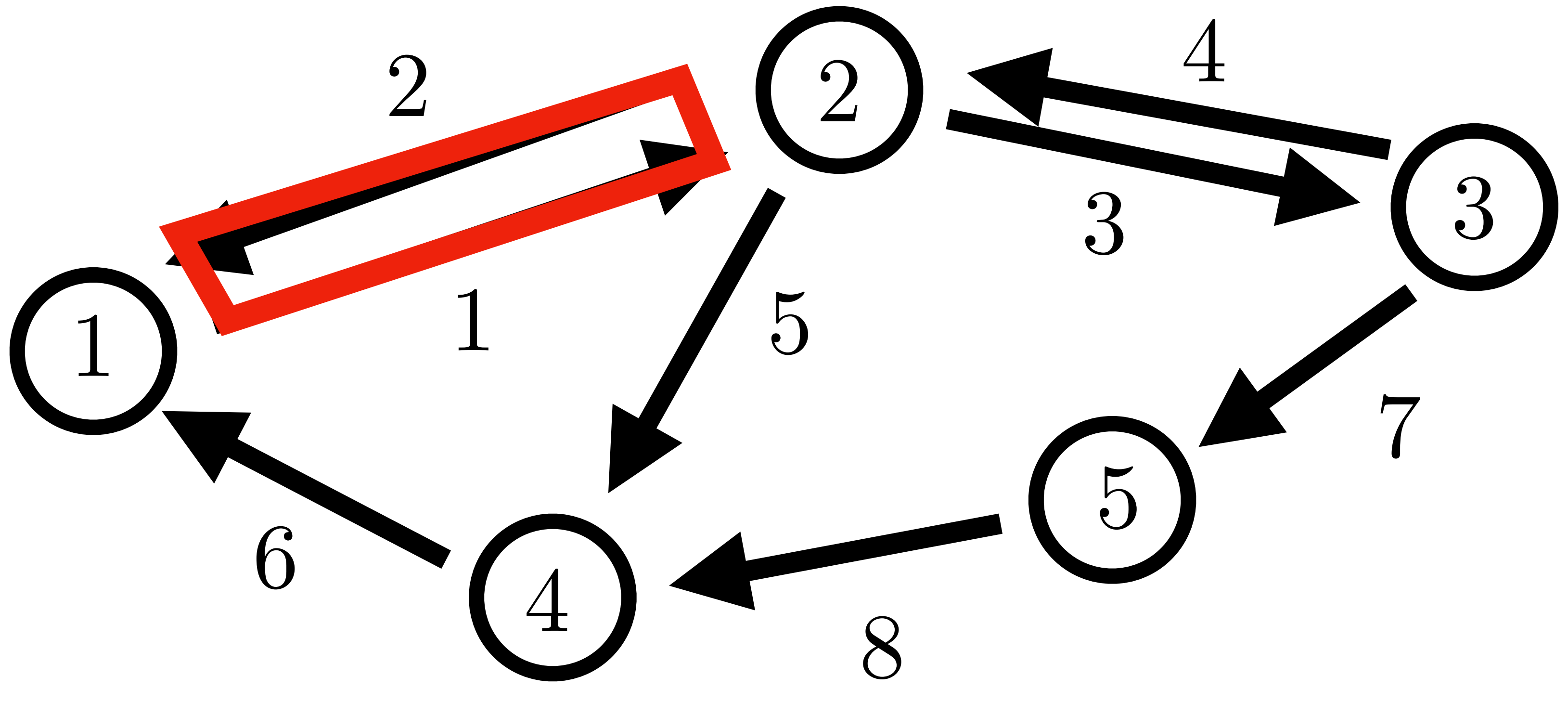
$$D_{\text{in}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Left Nullspace (General)

$$\begin{bmatrix} \mathbf{1}^T & 0 & \cdots & 0 \\ 0 & \mathbf{1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}^T \end{bmatrix} \Bigg[\ D\ \Bigg] = \mathbf{0}$$

dim left nullspace = num connected components

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, D \,\big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

edge mass flows ⟶

ea. equation:

Conservation of flow at ea. node

Cycle indicator (basis) matrix:

$$\big[\, C \,\big]_{ec} = \begin{cases} 1 & ; \text{ if } e \text{ flows with cycle } c \\ -1 & ; \text{ if } e \text{ flows against cycle } c \\ 0 & ; \ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

⟵ Sign indicates if cycle goes with or against edge direction

Columns = basis for $\mathcal{C}$

Right Nullspace

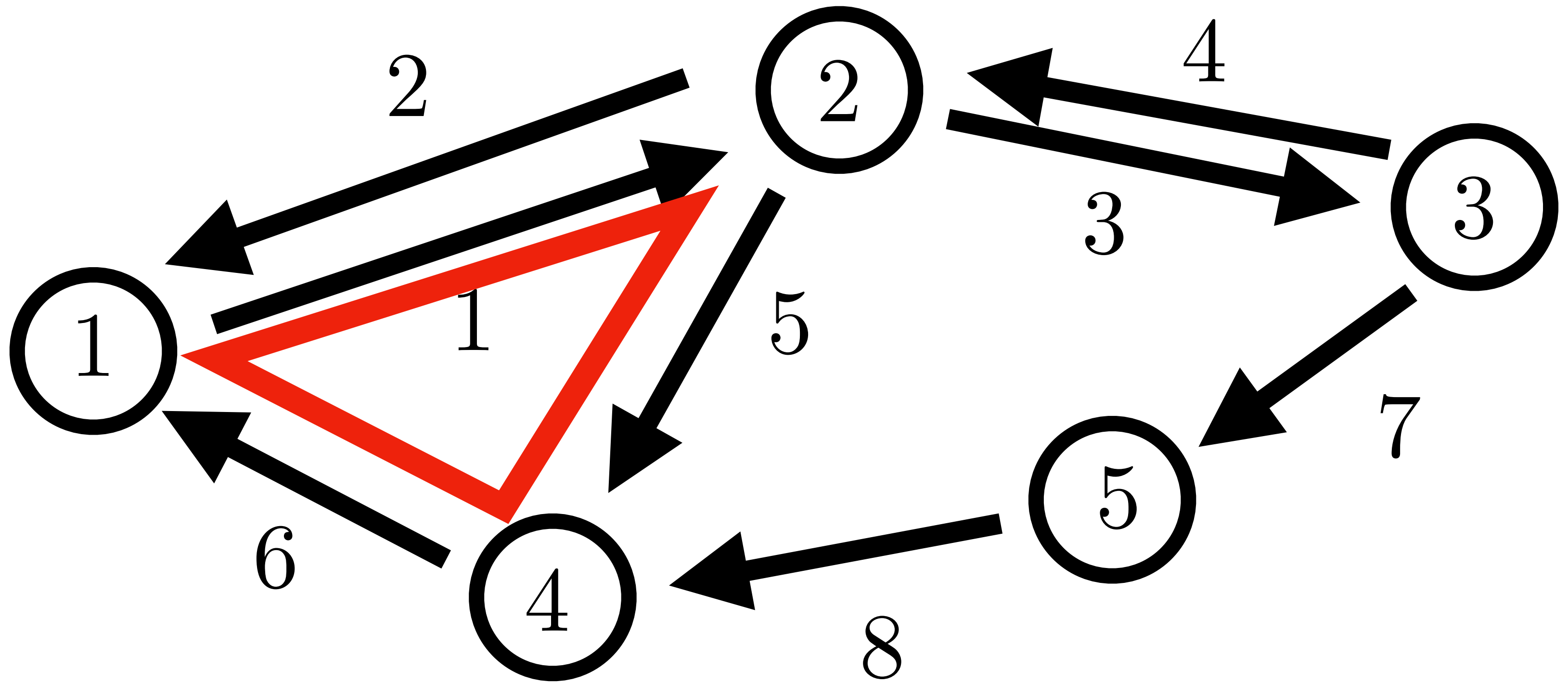
$$DC = 0$$

Cycle space $\mathcal{C}$

$$\dim\, \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$$

Ex: $8 - 5 + 1 = 4$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, D \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

ea. equation:

Conservation
of flow
at ea. node

edge mass flows ⟶

Right
Nullspace

$$DC = 0$$

Cycle space $\mathcal{C}$ $\quad \dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$

Ex: $8 - 5 + 1 = 4$

Cycle indicator (basis) matrix:

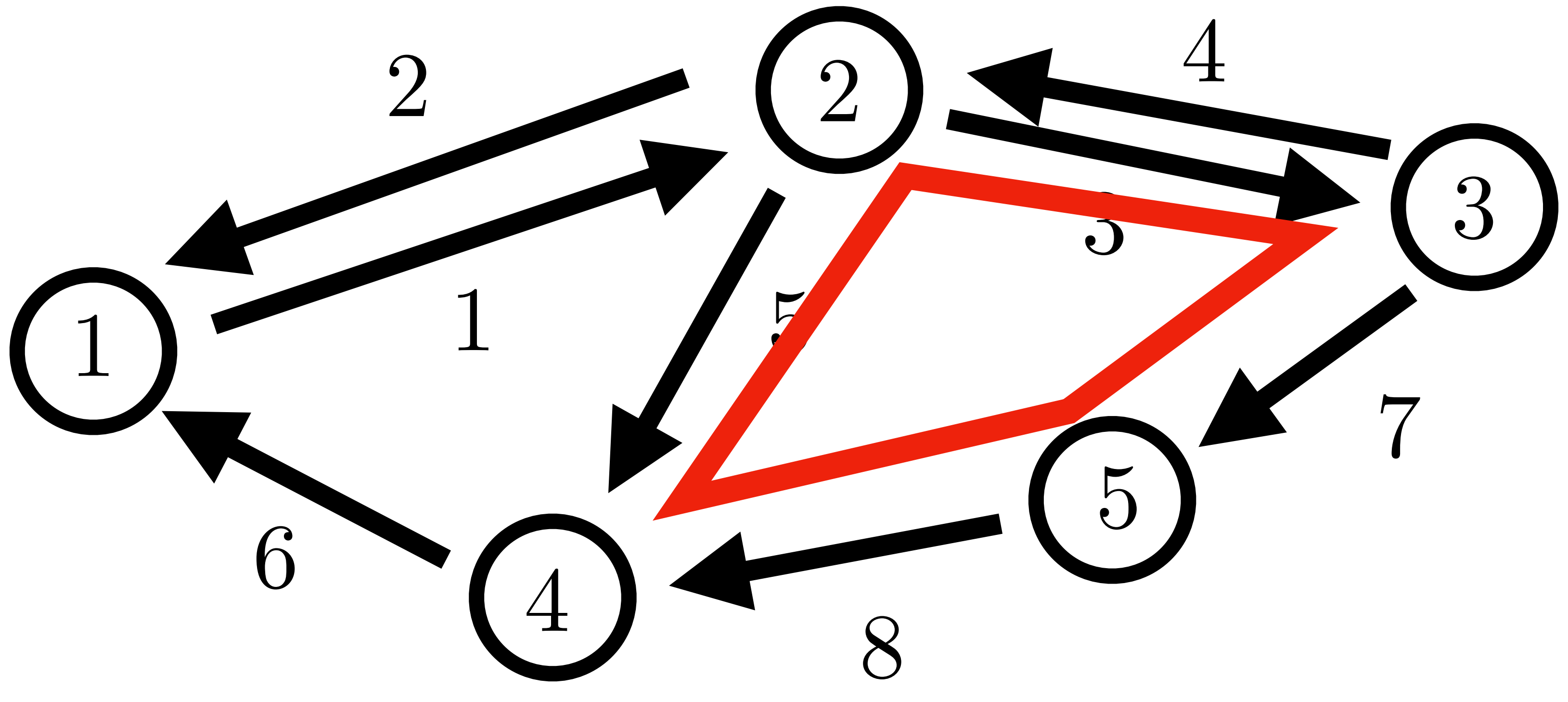
$$\left[\, C \,\right]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

⟵ Sign indicates
if cycle goes
with or against
edge direction

Columns = basis for $\mathcal{C}$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\, D \,\right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

edge mass flows ⟶

ea. equation:

Conservation
of flow
at ea. node

Cycle indicator (basis) matrix:

$$\left[\, C \,\right]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

Right
Nullspace

$$DC = 0$$

Cycle space $\mathcal{C}$

$$\dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$$

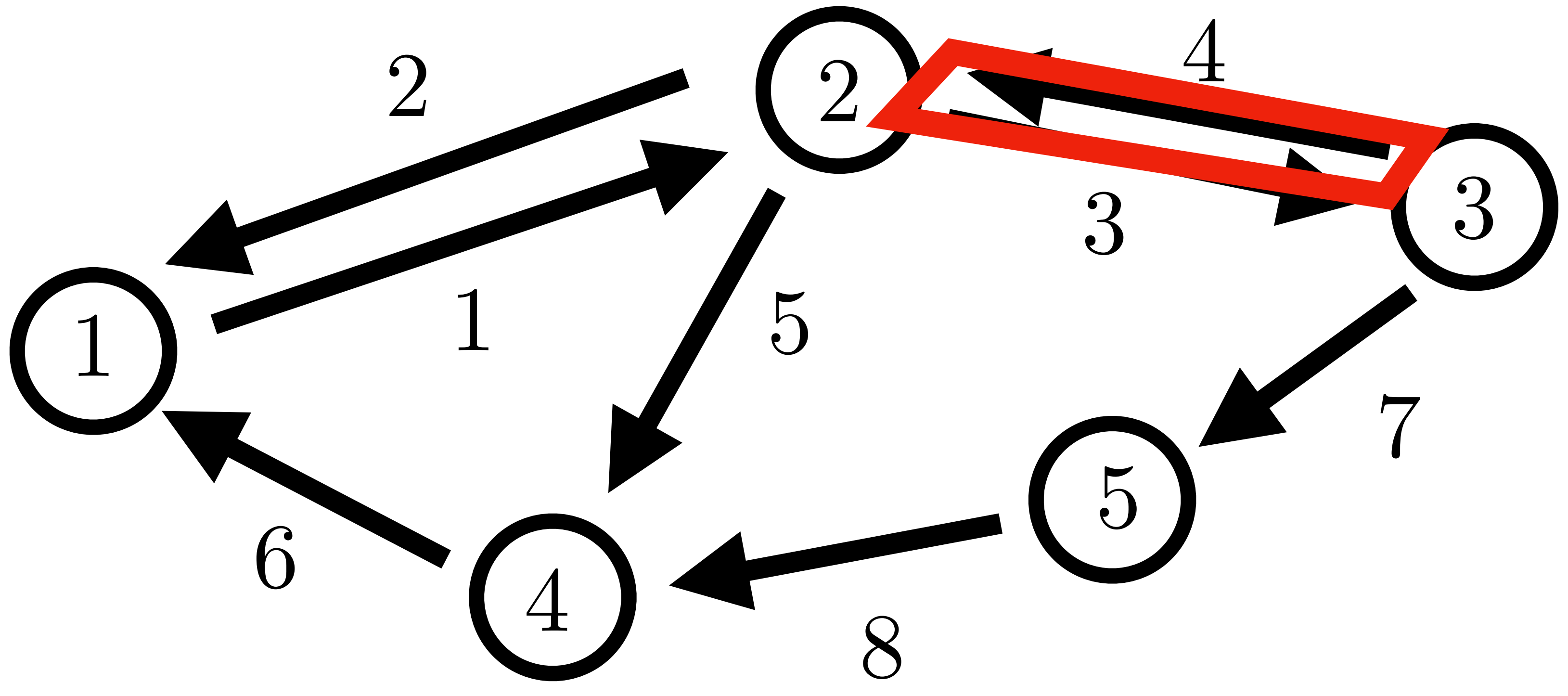
Ex: $8 - 5 + 1 = 4$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

⟵ Sign indicates
if cycle goes
with or against
edge direction

Columns =
basis for $\mathcal{C}$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, D \,\big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

edge mass flows ⟶

ea. equation:

Conservation
of flow
at ea. node

Cycle indicator (basis) matrix:

$$\big[\, C \,\big]_{ec} = \begin{cases} 1 & ; \text{ if } e \text{ flows with cycle } c \\ -1 & ; \text{ if } e \text{ flows against cycle } c \\ 0 & ; \ e \text{ not in cycle } c \end{cases}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

⟵ Sign indicates
if cycle goes
with or against
edge direction

Columns = basis for $\mathcal{C}$

Right
Nullspace

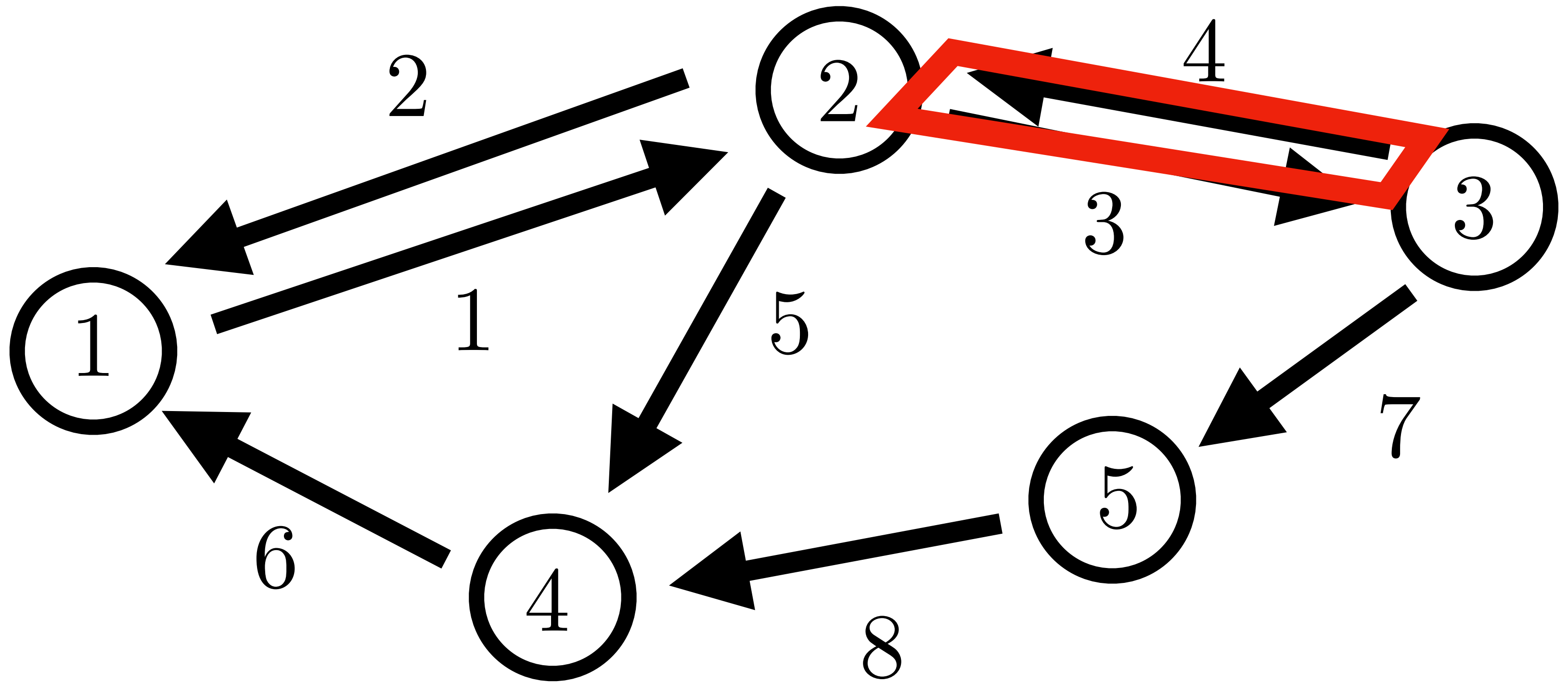
$$DC = 0$$

Cycle space $\mathcal{C}$

$$\dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$$

Ex: $8 - 5 + 1 = 4$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\left[\ D\ \right]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

edge mass flows ⟶

ea. equation:

Conservation
of flow
at ea. node

Cycle indicator (basis) matrix:

$$\left[\ C\ \right]_{ec} = \begin{cases} 1 & ;\ \text{if } e \text{ flows with cycle } c \\ -1 & ;\ \text{if } e \text{ flows against cycle } c \\ 0 & ;\ e \text{ not in cycle } c \end{cases}$$

Subspace
Constraint

$$Dx = 0 \quad \Rightarrow \quad x = Cz$$

Cycle space $\mathcal{C}$

$$\dim \mathcal{C} = |\mathcal{E}| - |\mathcal{V}| + 1$$

Ex: $8 - 5 + 1 = 4$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

⟵ Sign indicates
if cycle goes
with or against
edge direction

Columns = basis for $\mathcal{C}$

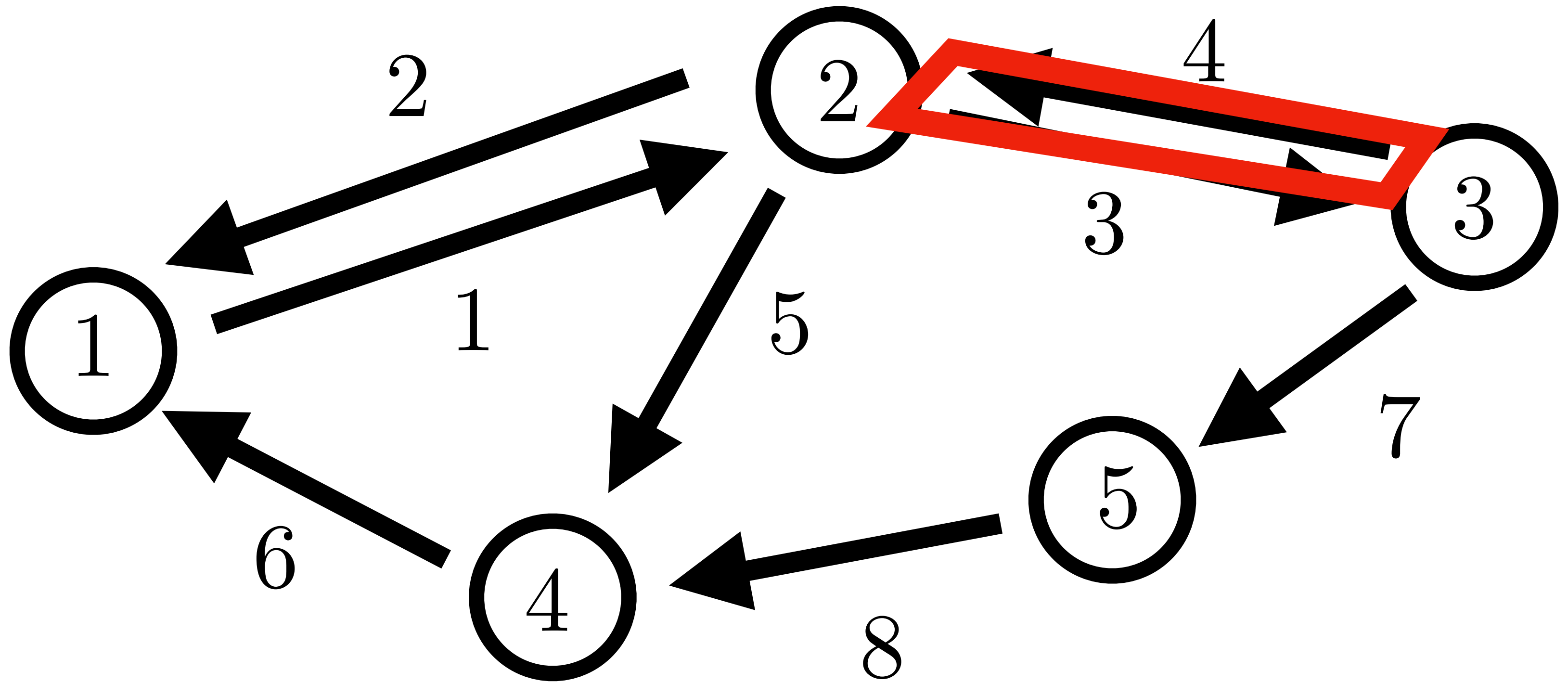
Incidence Matrix - Right Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad \begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$



$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Cycle indicator (basis) matrix:

$$\begin{bmatrix} C \end{bmatrix}_{ec} = \begin{cases} 1 & ; \text{ if } e \text{ flows with cycle } c \\ -1 & ; \text{ if } e \text{ flows against cycle } c \\ 0 & ; e \text{ not in cycle } c \end{cases}$$

Affine Constraint

$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

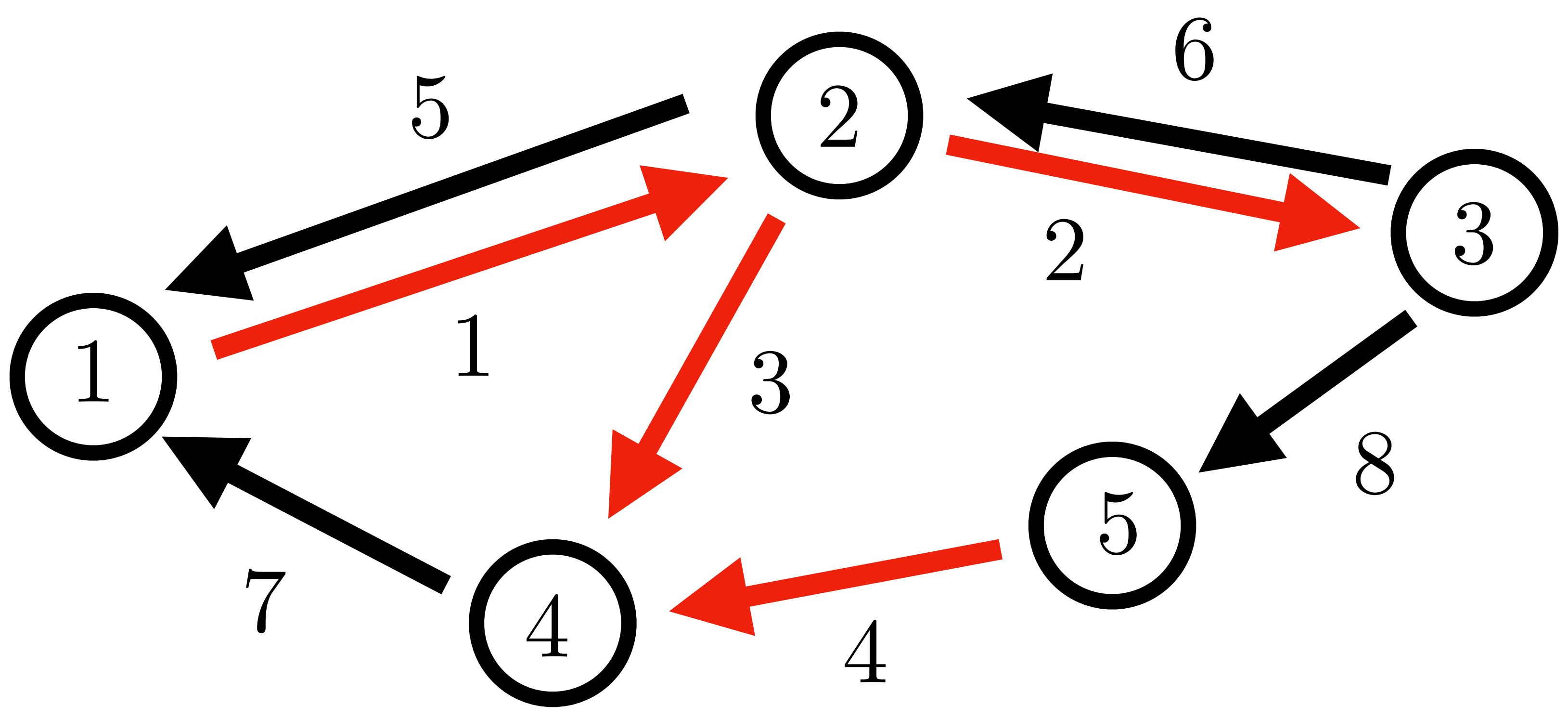
$$\bar{x} = D^T (DD^T)^{\dagger} S$$

$$C = \begin{bmatrix} 1 & 1 & 0 & \boxed{0} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

Sign indicates if cycle goes with or against edge direction

Columns = basis for $\mathcal{C}$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows →

Affine Constraint

$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^{\dagger} S$$

Spanning Tree Construction:

$$D = \big[D' \;\; D''\big] = \big[D' \;\; D'M\big] = D'\big[I \;\; M\big]$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|\mathcal{V}| - 1$$

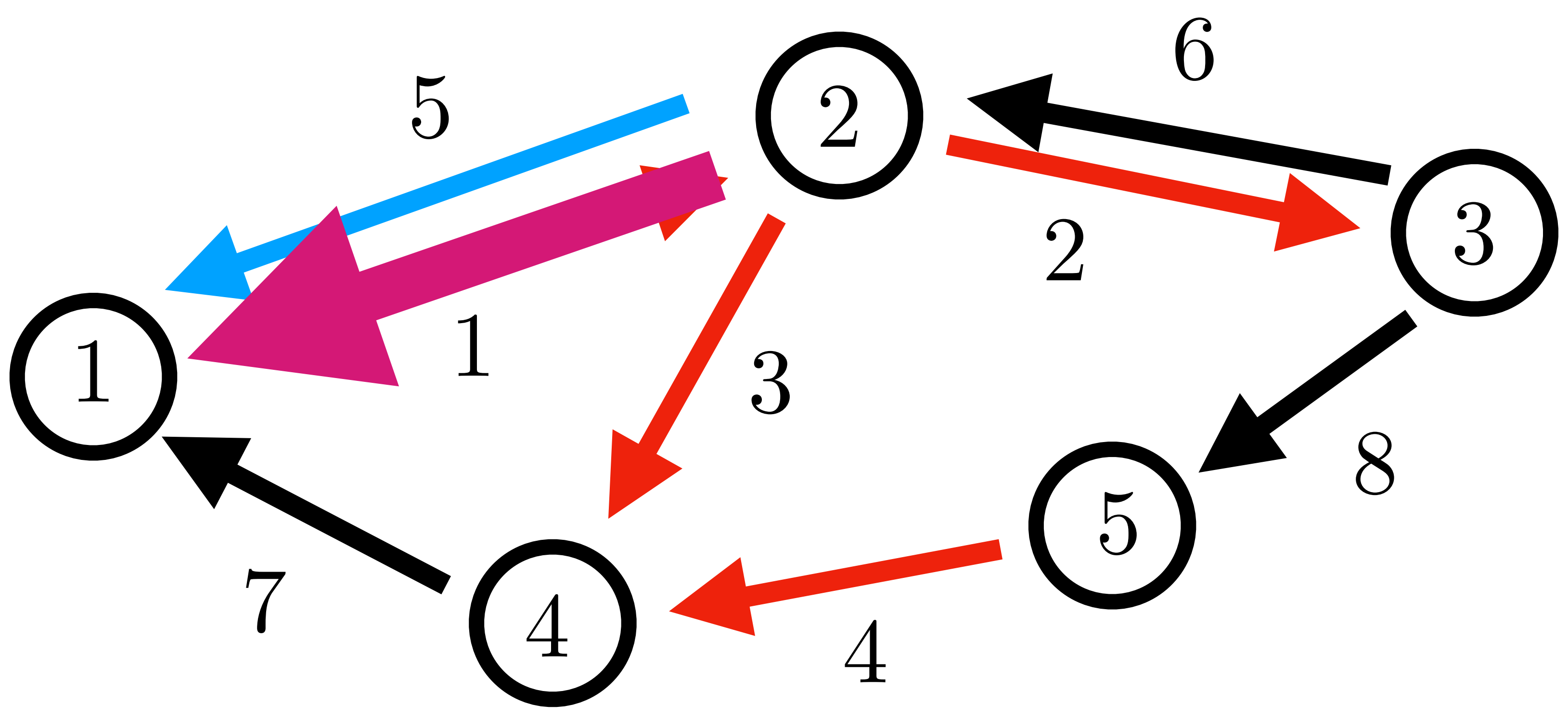
Incidence Matrix - Right Nullspace

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$



$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$\begin{array}{cc} |\mathcal{V}| & \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \\ & |\mathcal{V}| - 1 \end{array}$$

Affine Constraint

$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

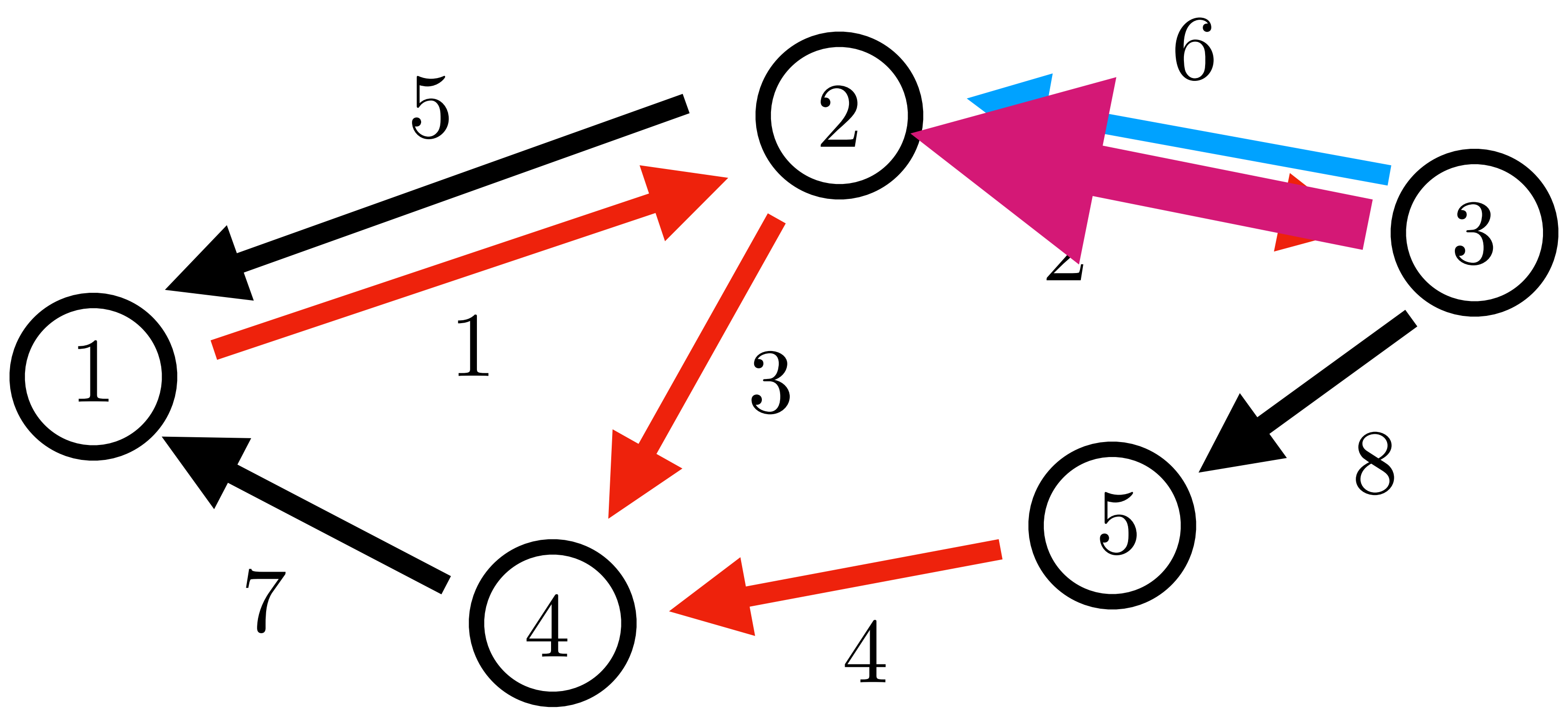
Cyclic Flow

$$D'M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Min Norm Solution

$$\bar{x} = D^T (DD^T)^\dagger S$$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^\dagger S$$

Spanning Tree Construction:

$$D = \big[D' \quad D''\big] = \big[D' \quad D'M\big] = D'\big[I \quad M\big]$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|\mathcal{V}| - 1$$

$$D'M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Incidence Matrix - Right Nullspace

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

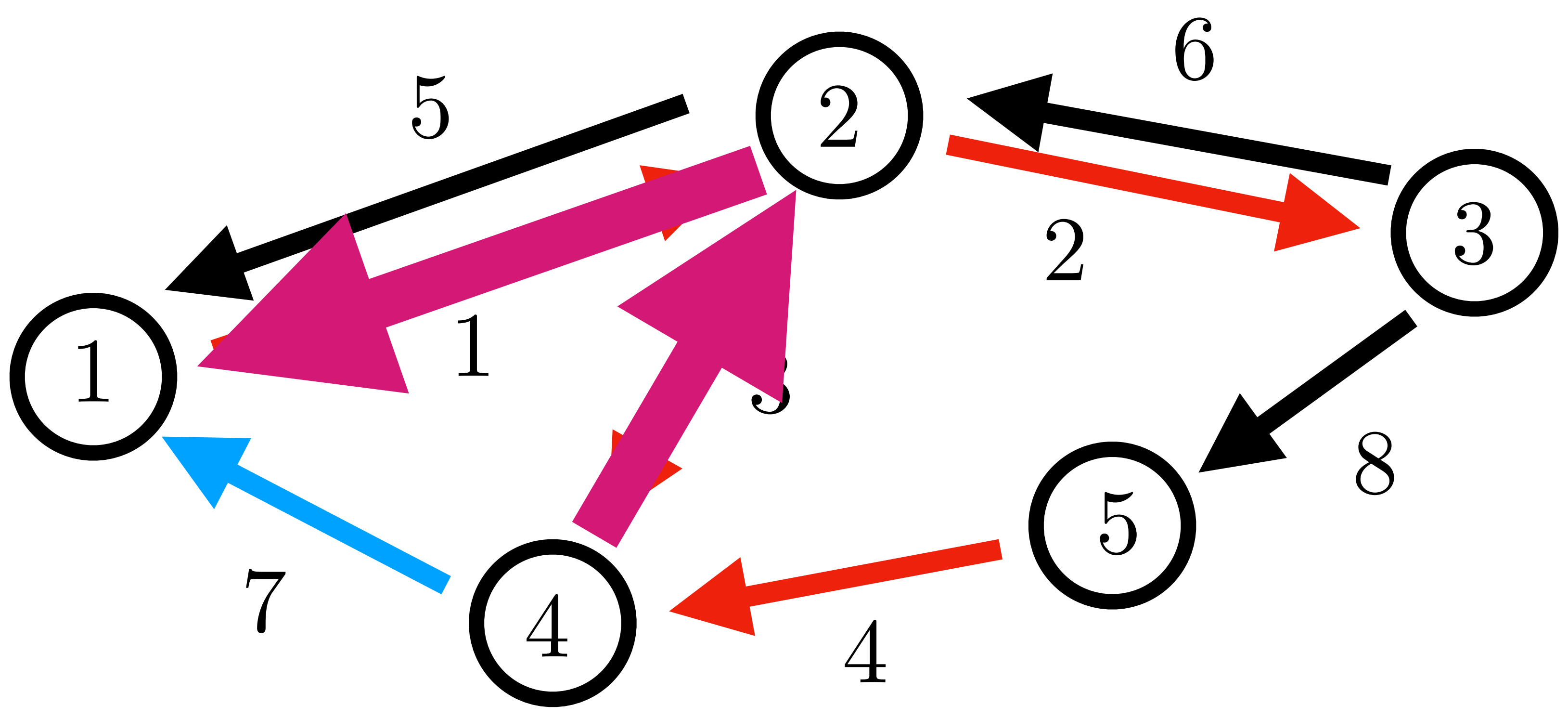
$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^{\dagger} S$$



Spanning Tree Construction:

$$D = \big[D' \quad D''\big] = \big[D' \quad D'M\big] = D'\big[I \quad M\big]$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|\mathcal{V}| - 1$$

$$D'M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Incidence Matrix - Right Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$

$$\begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

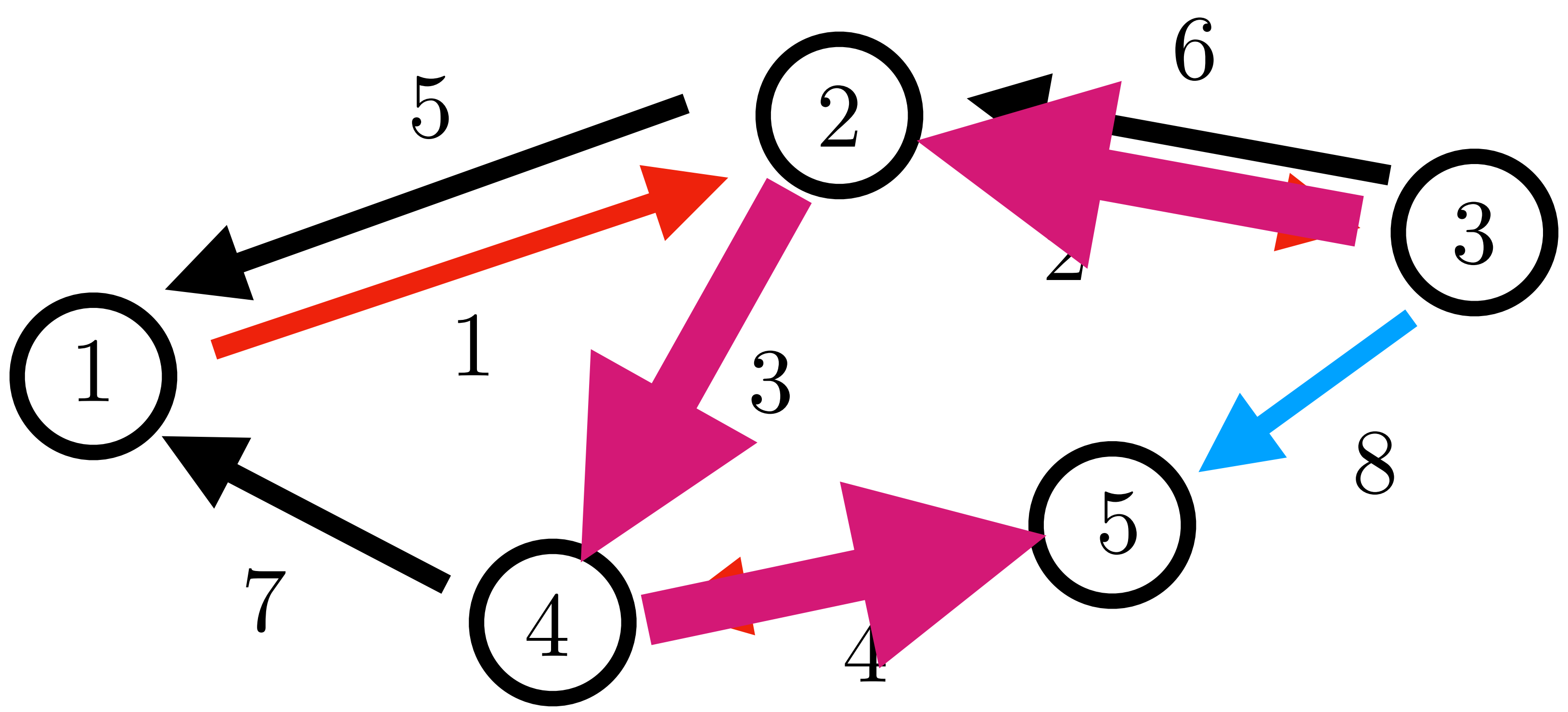
$$Dx = S \Rightarrow x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T(DD^T)^{\dagger}S$$



Spanning Tree Construction:

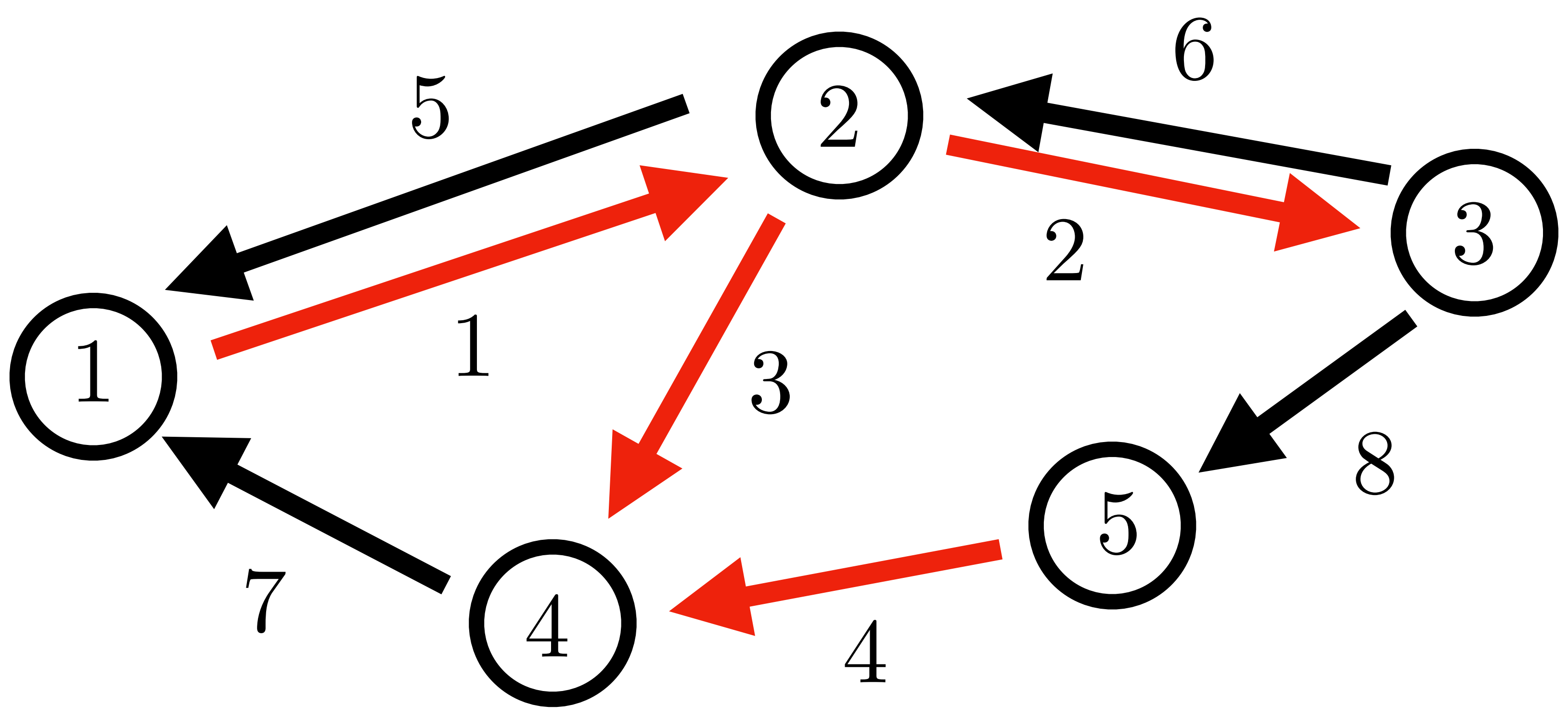
$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D'\begin{bmatrix} I & M \end{bmatrix}$$

$$|\mathcal{V}| \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|\mathcal{V}| - 1$$

$$D'M = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad \begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

$$Dx = S \ \Rightarrow \ x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^{\dagger} S$$

Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \left[\begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right]$$

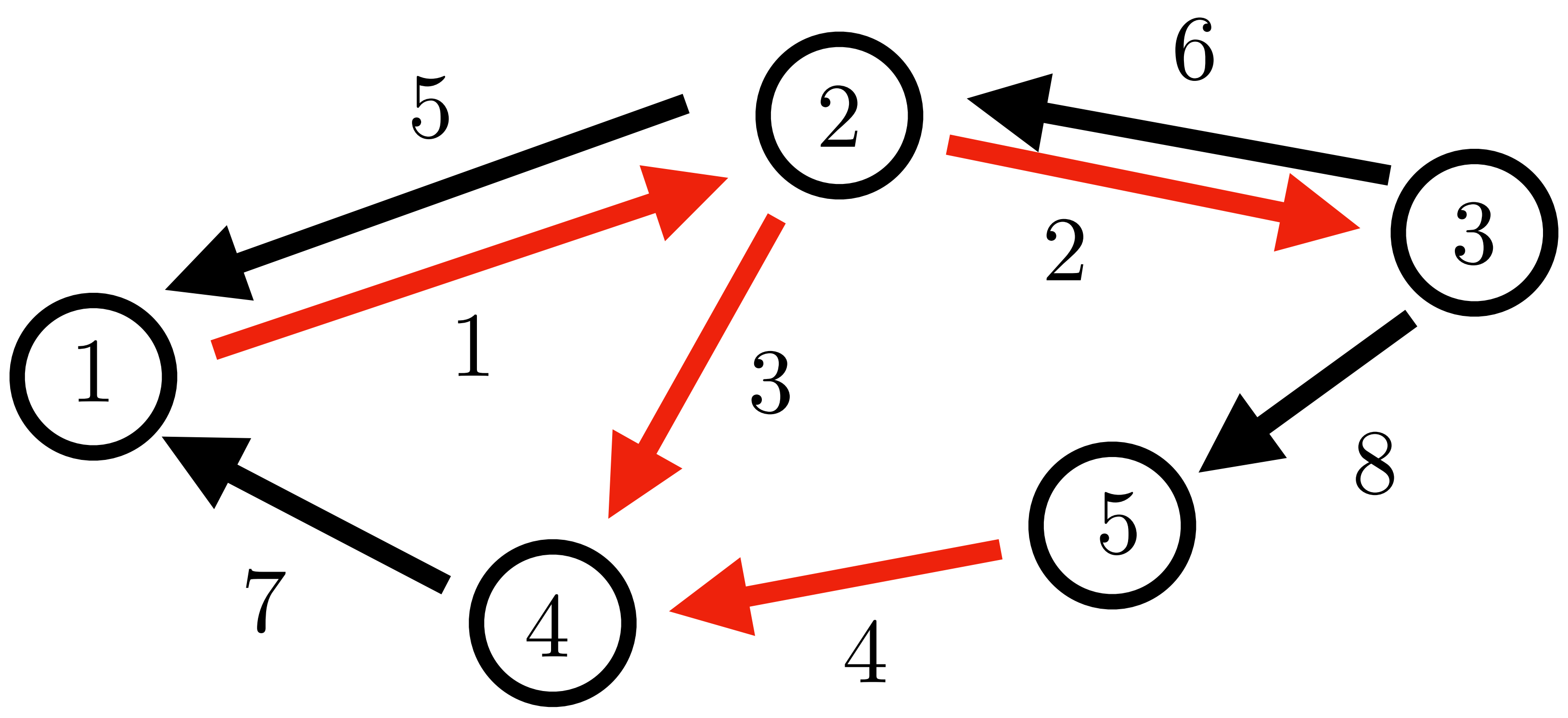
Incidence Matrix - Right Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad \Big[\, D \,\Big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$



$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows →

Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Affine Constraint

$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

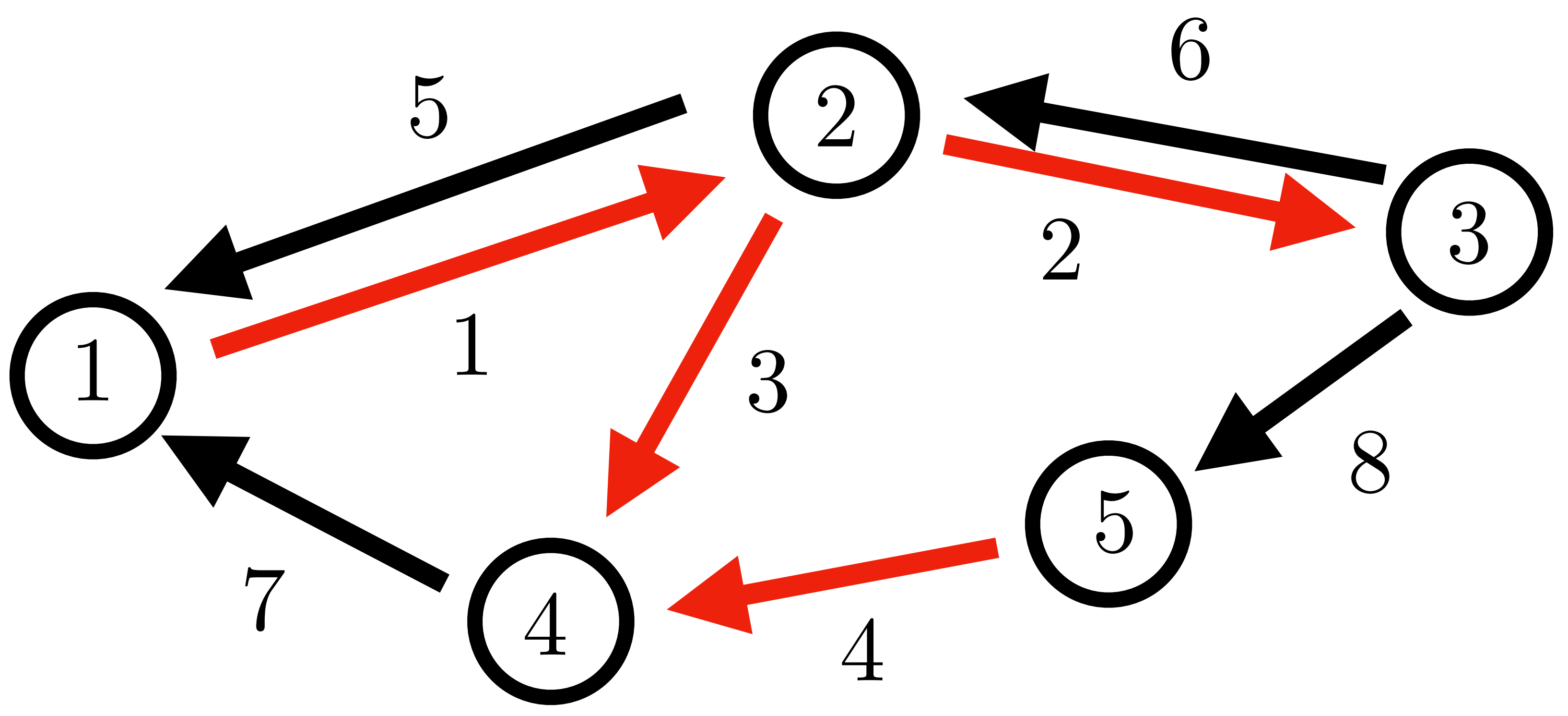
Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^{\dagger} S$$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $\quad \begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & \text{; if } e \text{ out of } v \\ 1 & \text{; if } e \text{ into } v \\ 0 & \text{; otherwise} \end{cases}$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows →

Affine Constraint $\quad Dx = S \Rightarrow x = \bar{x} + Cz$

Specific Solution

Cyclic Flow

Min Norm Solution $\quad \bar{x} = D^T (DD^T)^\dagger S$

Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} D' & D'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = D'M - D'' = 0$$

Incidence Matrix - Right Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

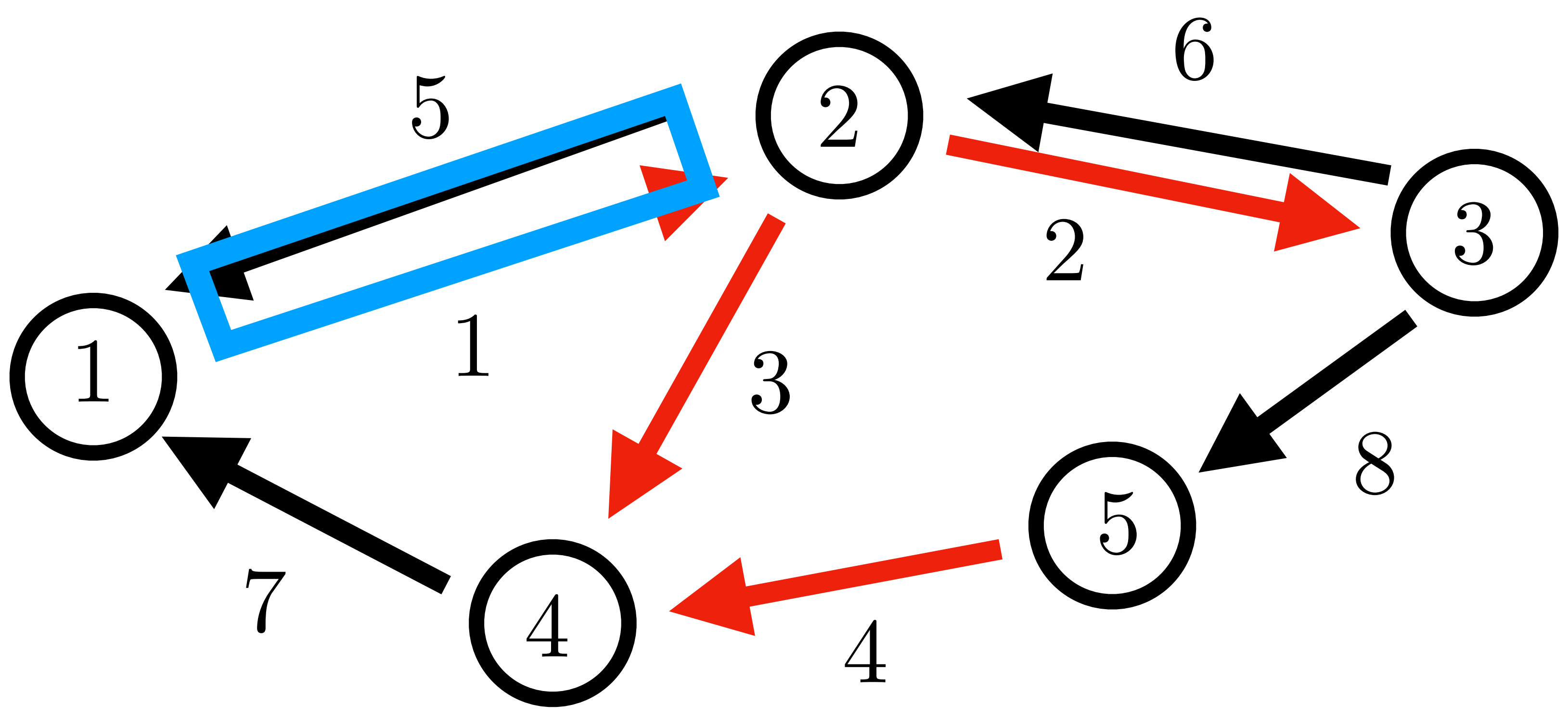
$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^{\dagger} S$$



Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} D' & D'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = D'M - D'' = 0$$

Incidence Matrix - Right Nullspace

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\begin{bmatrix} D \end{bmatrix}_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

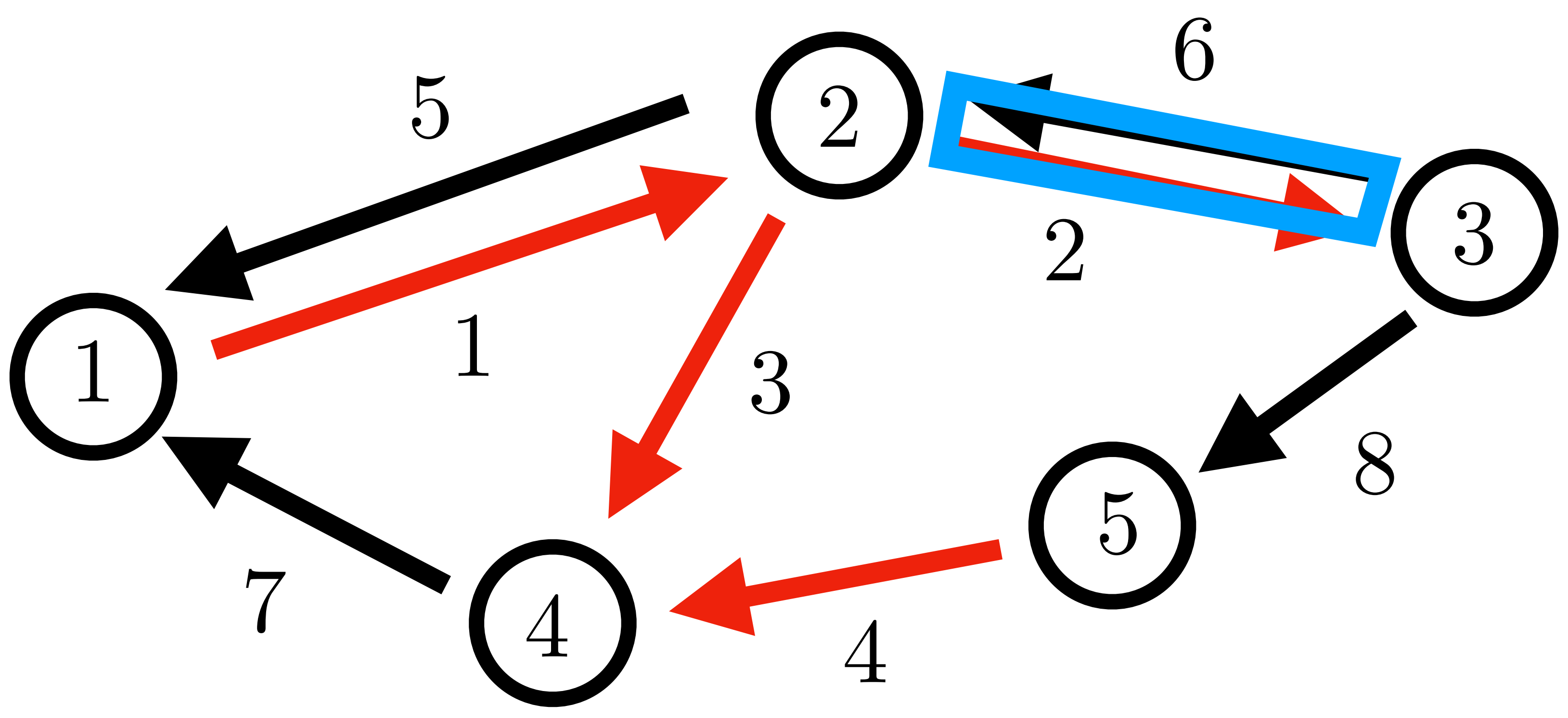
$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^\dagger S$$



Spanning Tree Construction:

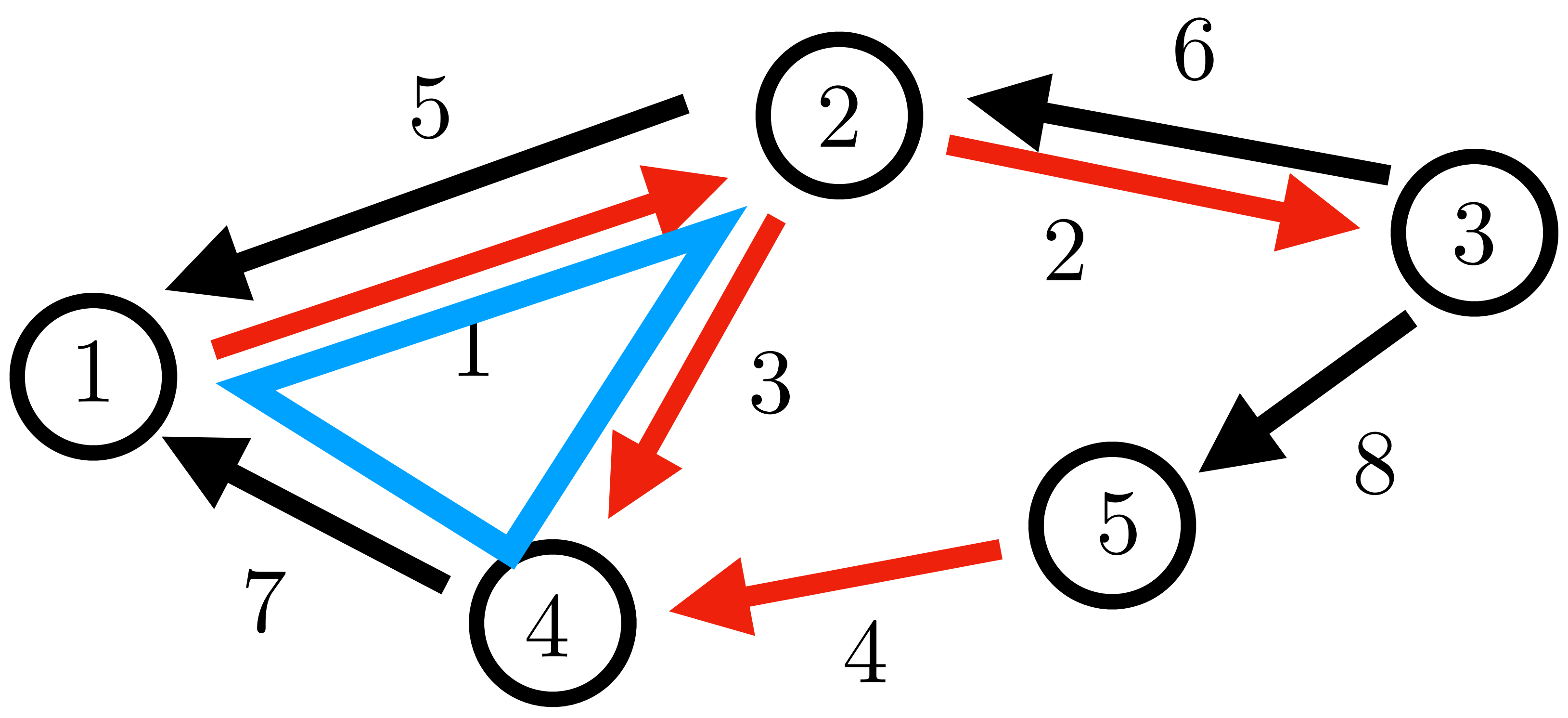
$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} D' & D'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = D'M - D'' = 0$$

Incidence Matrix - Right Nullspace



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\big[\, D \,\big]_{ve} = \begin{cases} -1 & ;\ \text{if } e \text{ out of } v \\ 1 & ;\ \text{if } e \text{ into } v \\ 0 & ;\ \text{otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

$$Dx = S \ \Rightarrow \ x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^\dagger S$$

Spanning Tree Construction:

$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} D' & D'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = D'M - D'' = 0$$

Incidence Matrix - Right Nullspace

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$\Big[D \Big]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{bmatrix}$$

flow source at ea. node

edge mass flows ⟶

Affine Constraint

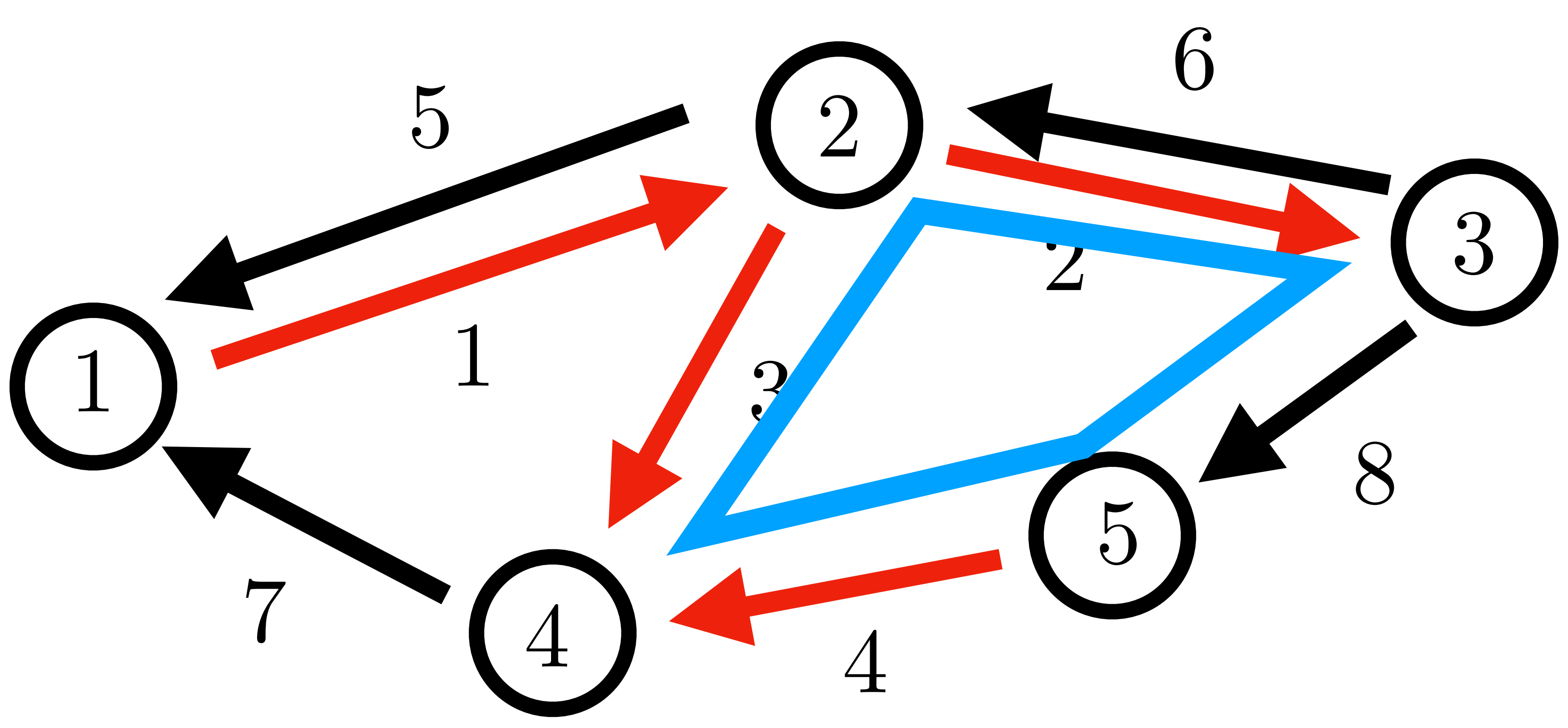
$$Dx = S \;\Rightarrow\; x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Min Norm Solution

$$\bar{x} = D^T (DD^T)^\dagger S$$



Spanning Tree Construction:

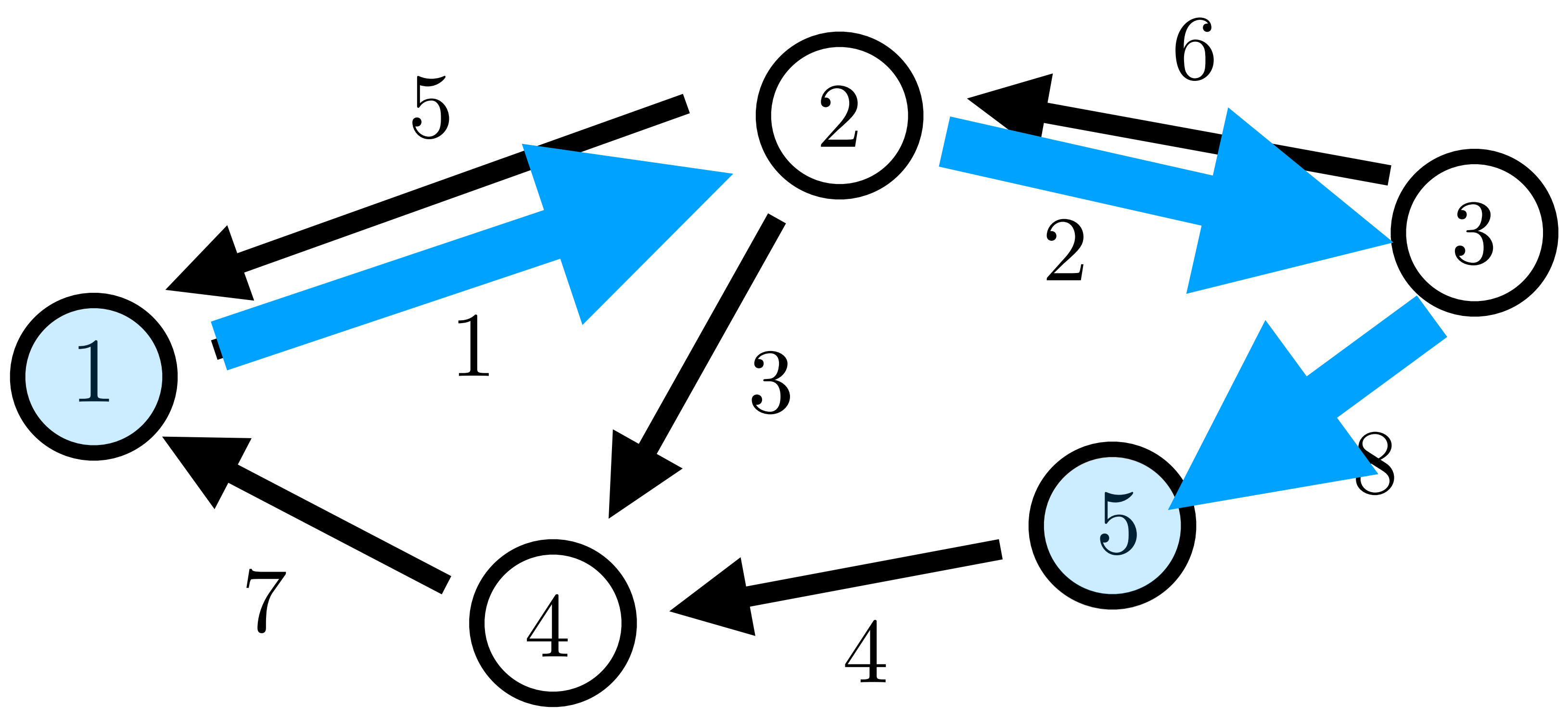
$$D = \begin{bmatrix} D' & D'' \end{bmatrix} = \begin{bmatrix} D' & D'M \end{bmatrix} = D' \begin{bmatrix} I & M \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} M \\ -I \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} D' & D'' \end{bmatrix} \begin{bmatrix} M \\ -I \end{bmatrix} = D'M - D'' = 0$$

Incidence Matrix - Column Geometry



Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

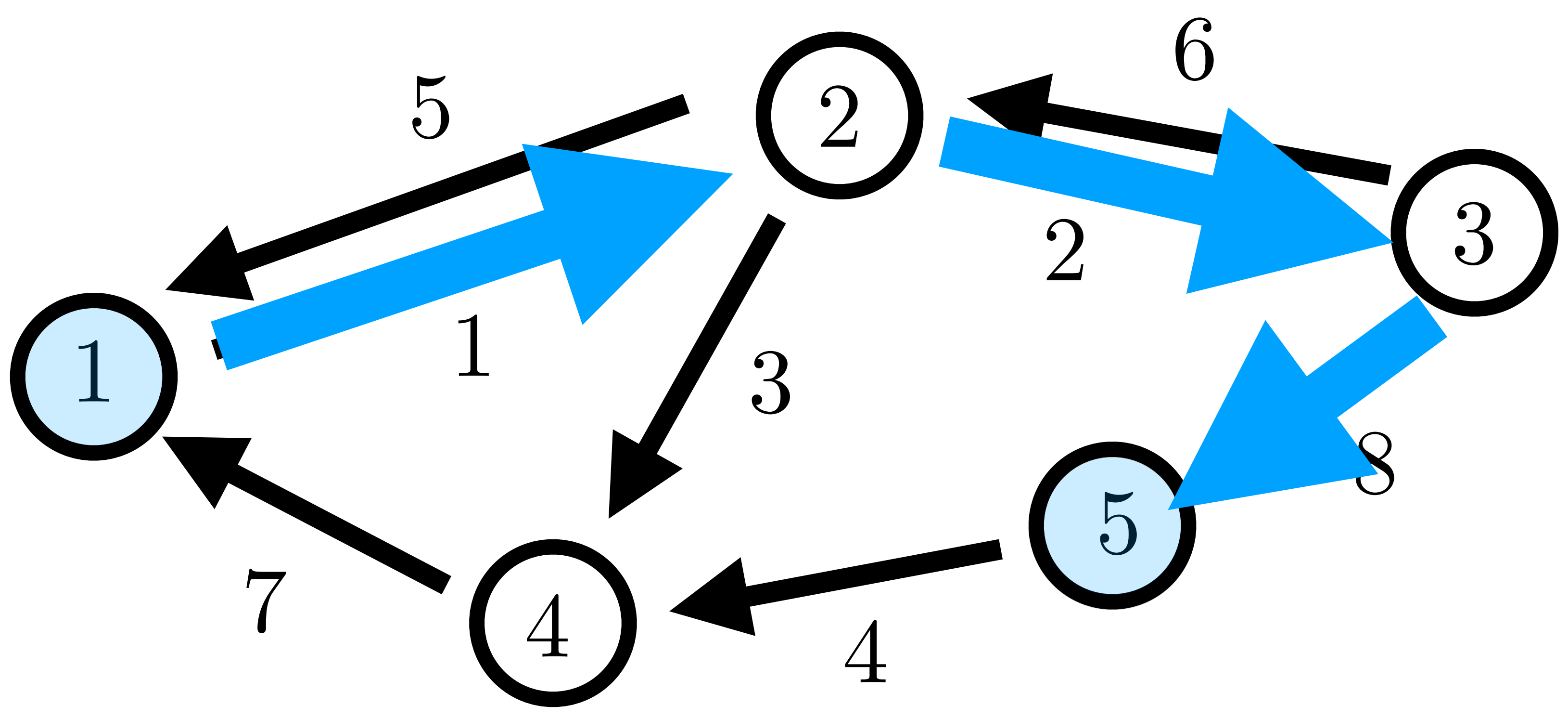
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Incidence Matrix - Column Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

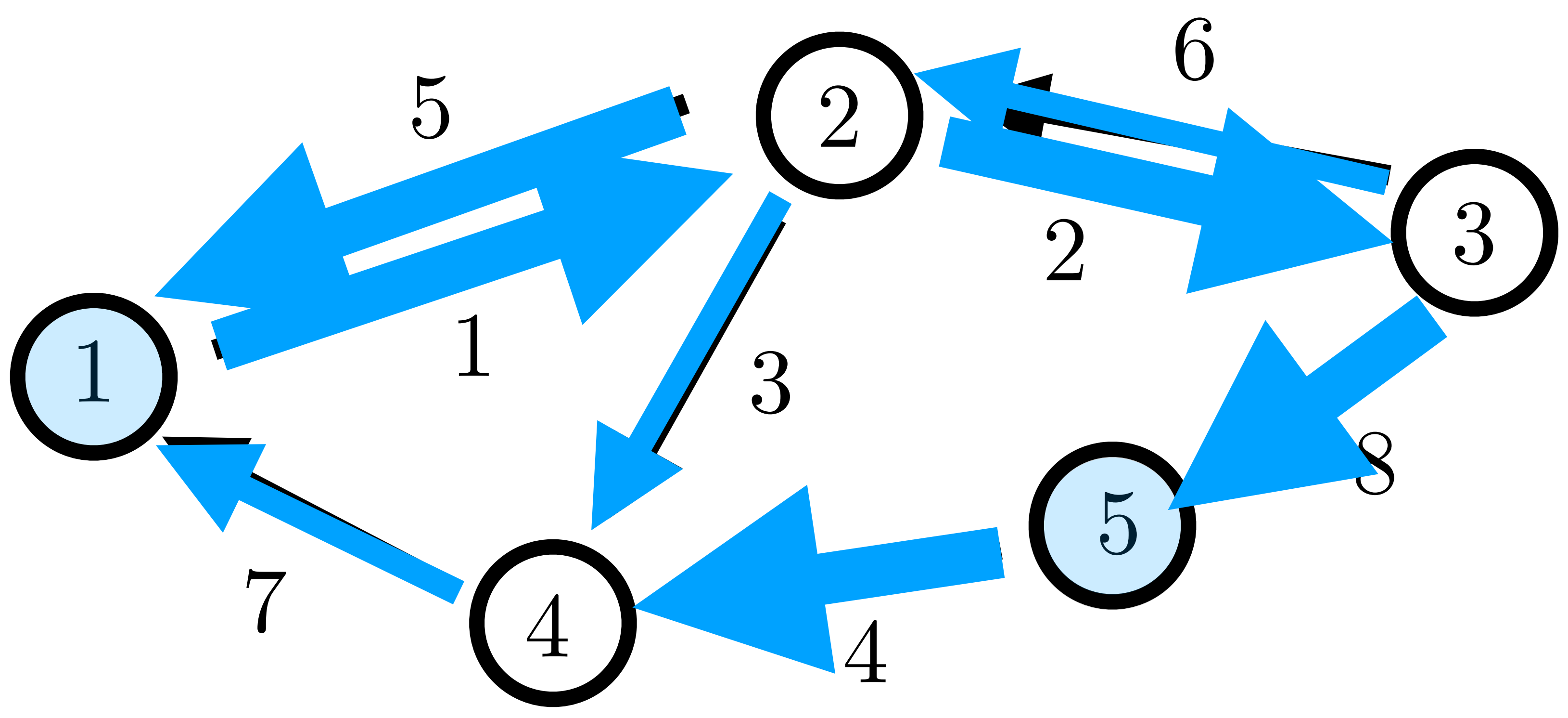
$$S = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Incidence Matrix - Column Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

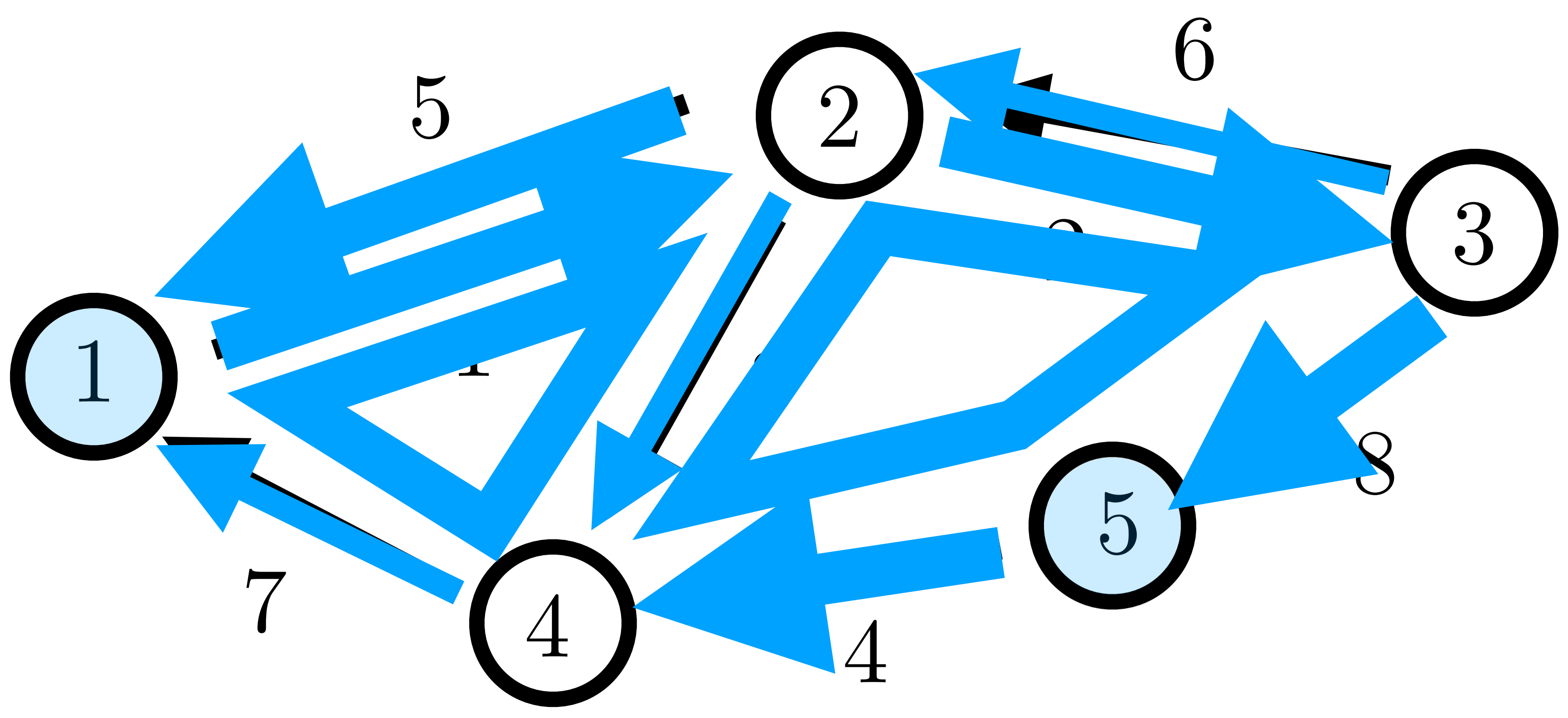
$$S = \begin{bmatrix} -1 \\ 0.5 \\ 2 \\ 0.2 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Incidence Matrix - Column Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

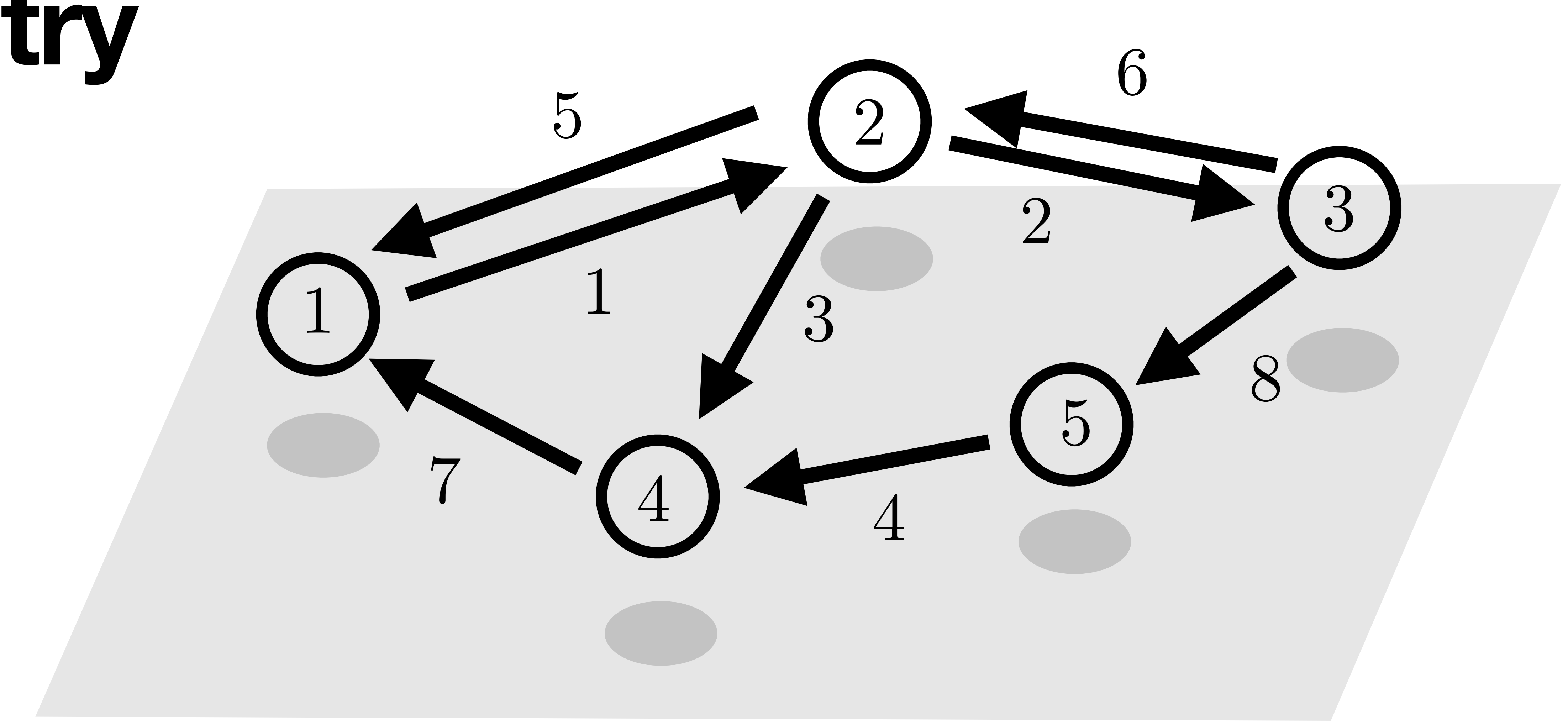
$$S = \begin{bmatrix} -1 \\ 0.5 \\ 2 \\ 0.2 \\ 1 \end{bmatrix}$$

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Incidence Matrix - Column Geometry



Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Incidence Matrix - Column Geometry

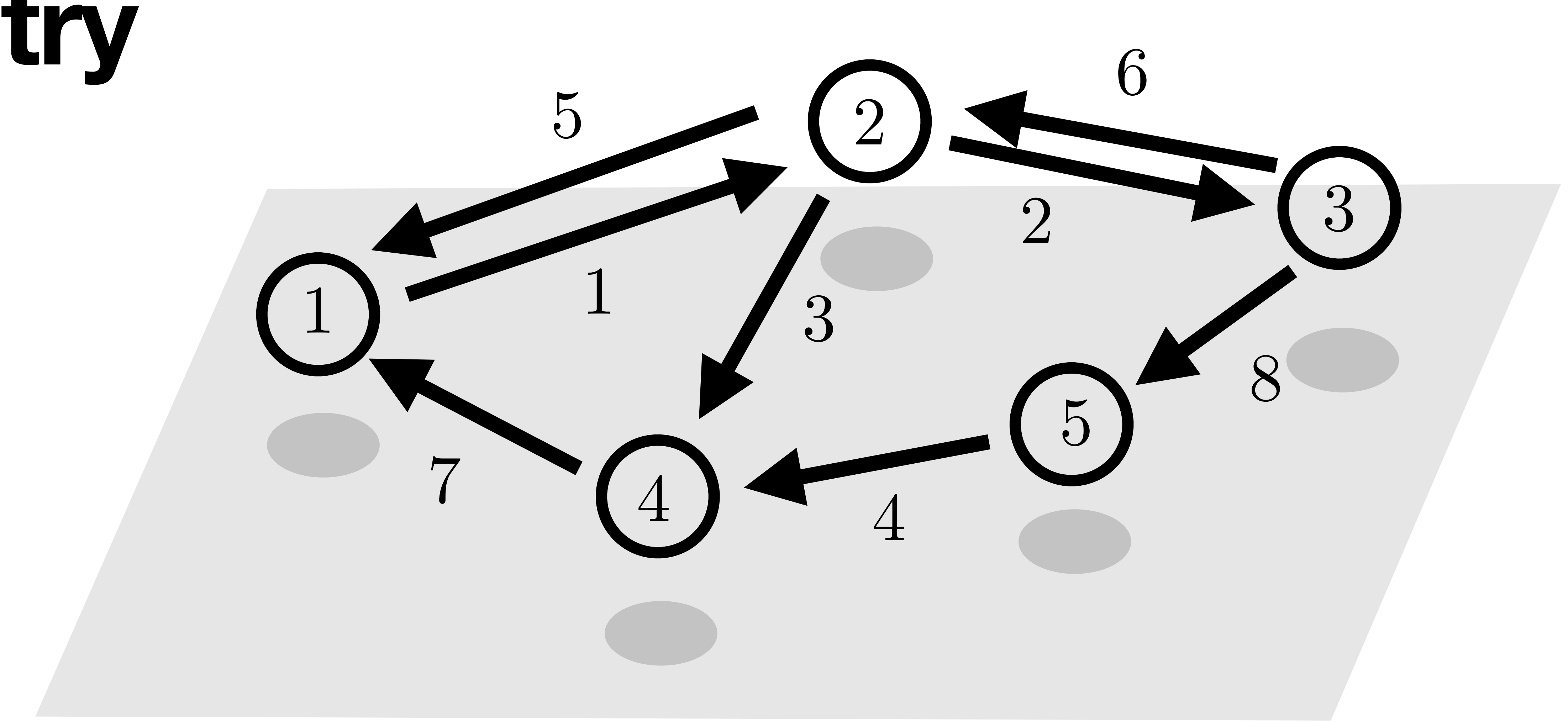
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

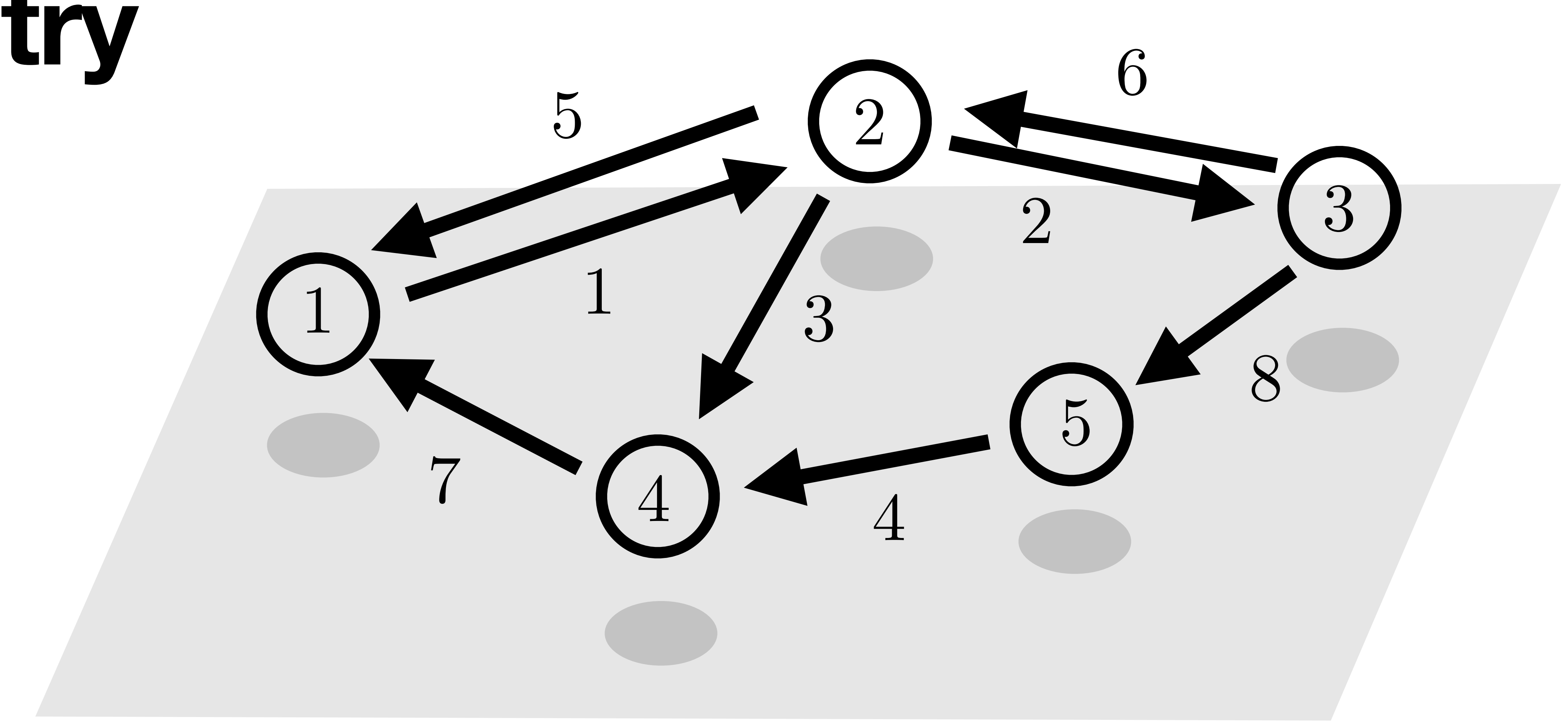
Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**
...value function on nodes

Non-conserved flow $\quad S = Dx \quad$ **Edge flow vector**

$$x = \bar{x} + Cz$$

Specific Solution $\quad$ Cyclic Flow

Incidence Matrix - Column Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**
...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

Value Function

Cost-to-go
Potential value
“Height” - gravitational potential

Incidence Matrix - Column Geometry

Graph:

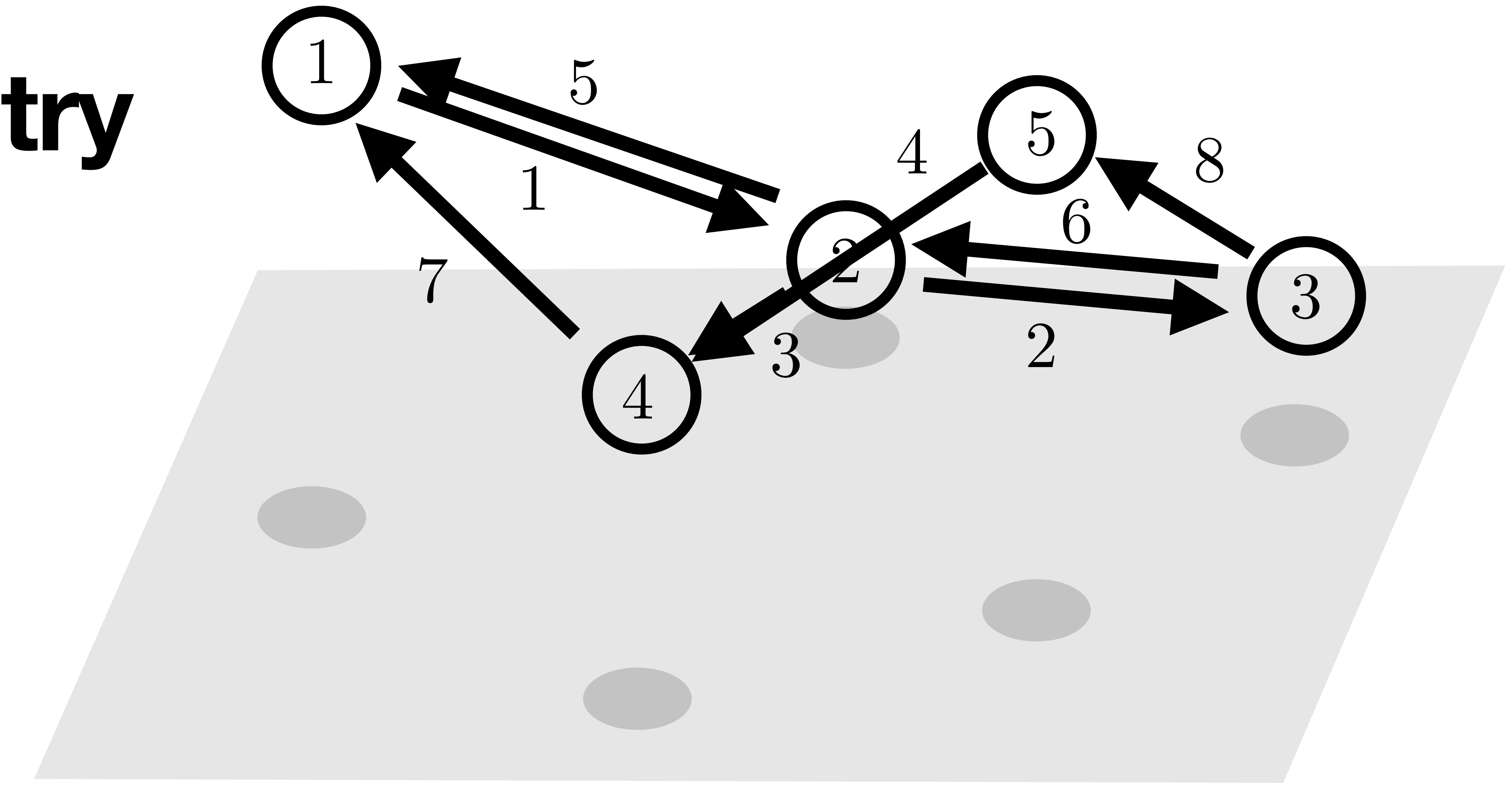
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes

...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

Value Function

Cost-to-go

Potential value

“Height” - gravitational potential

Incidence Matrix - Column Geometry

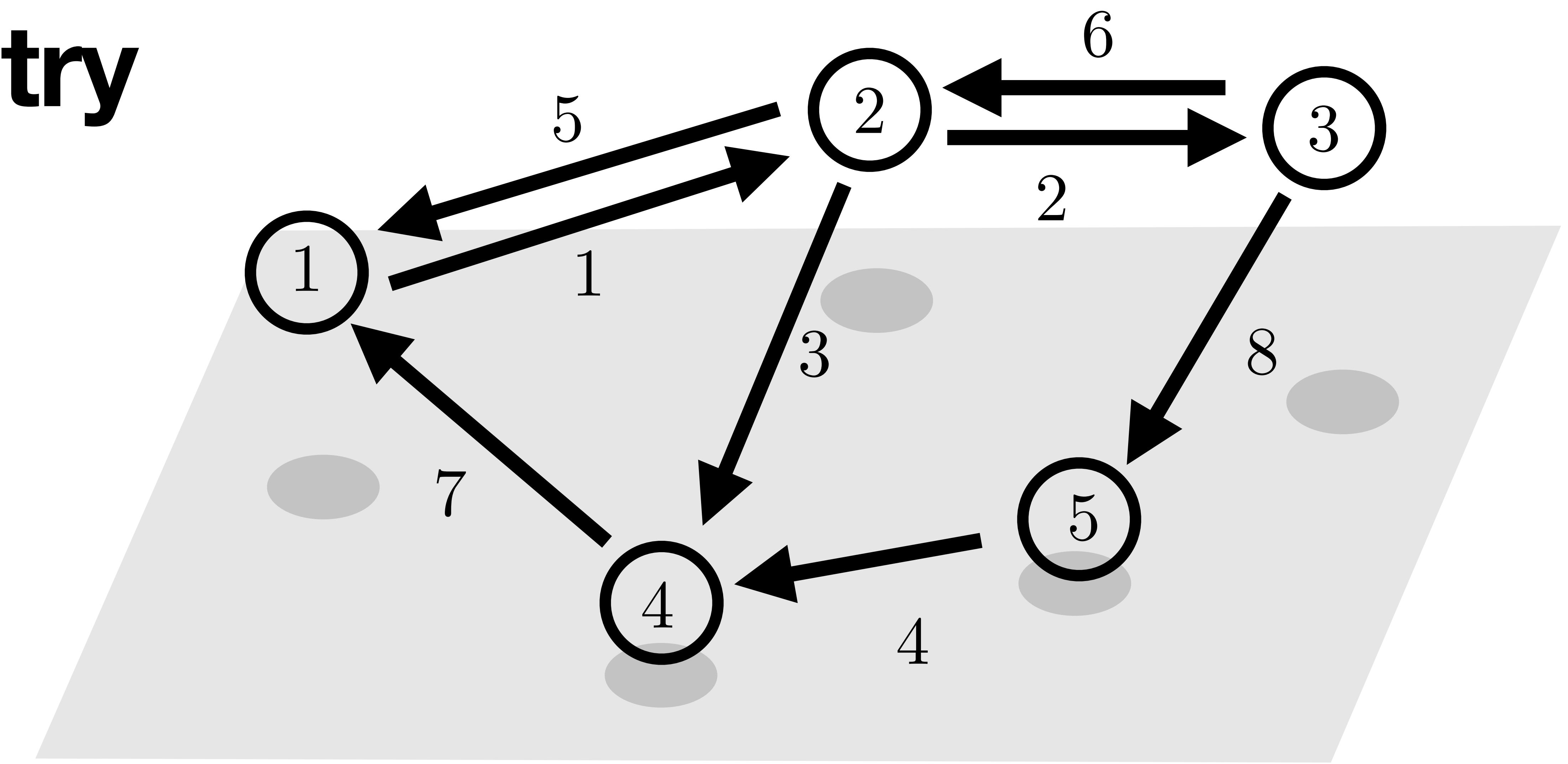
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes
...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

Value Function

Cost-to-go
Potential value
“Height” - gravitational potential

Incidence Matrix - Column Geometry

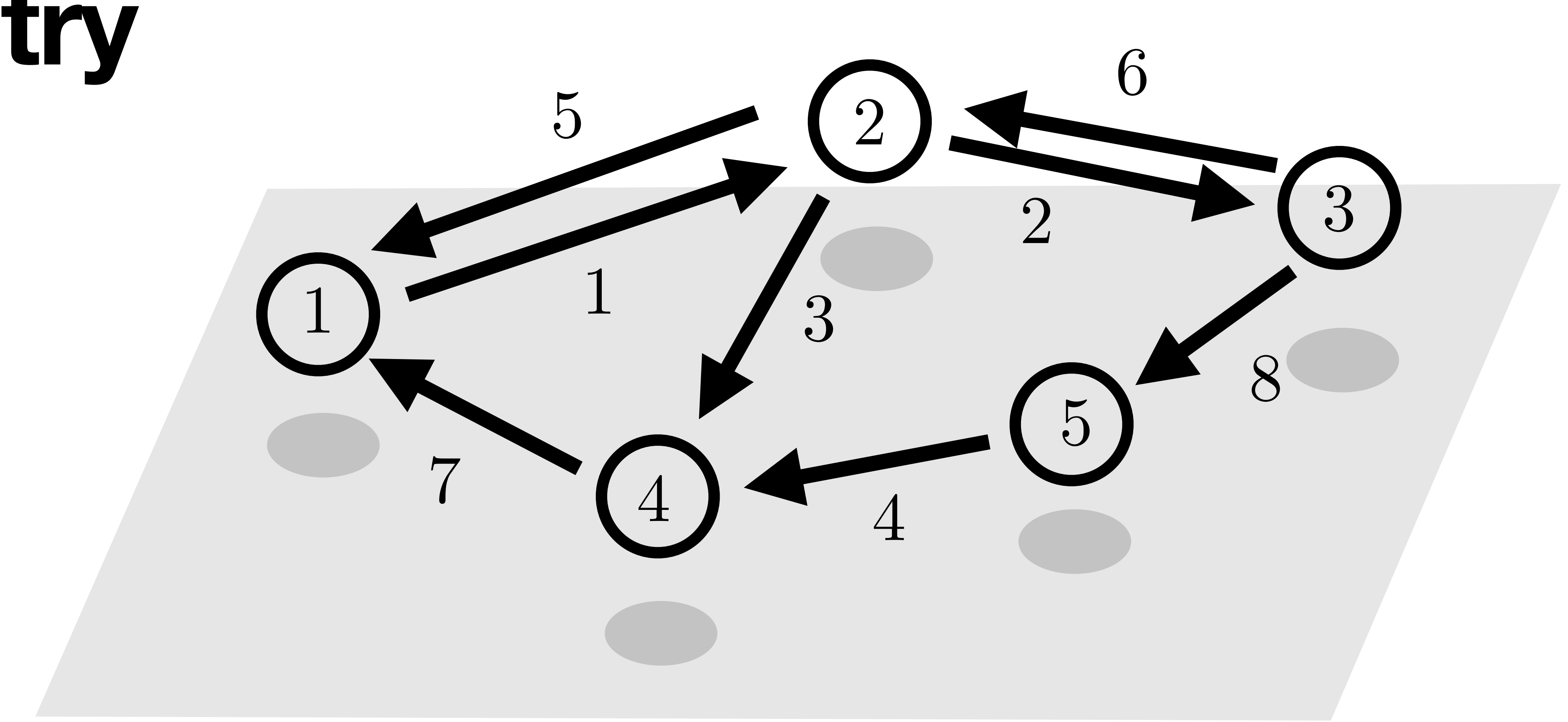
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes
...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

$$\left[w^T D\right]_e = w_i - w_{i'} = \tau_e$$

Incidence Matrix - Column Geometry

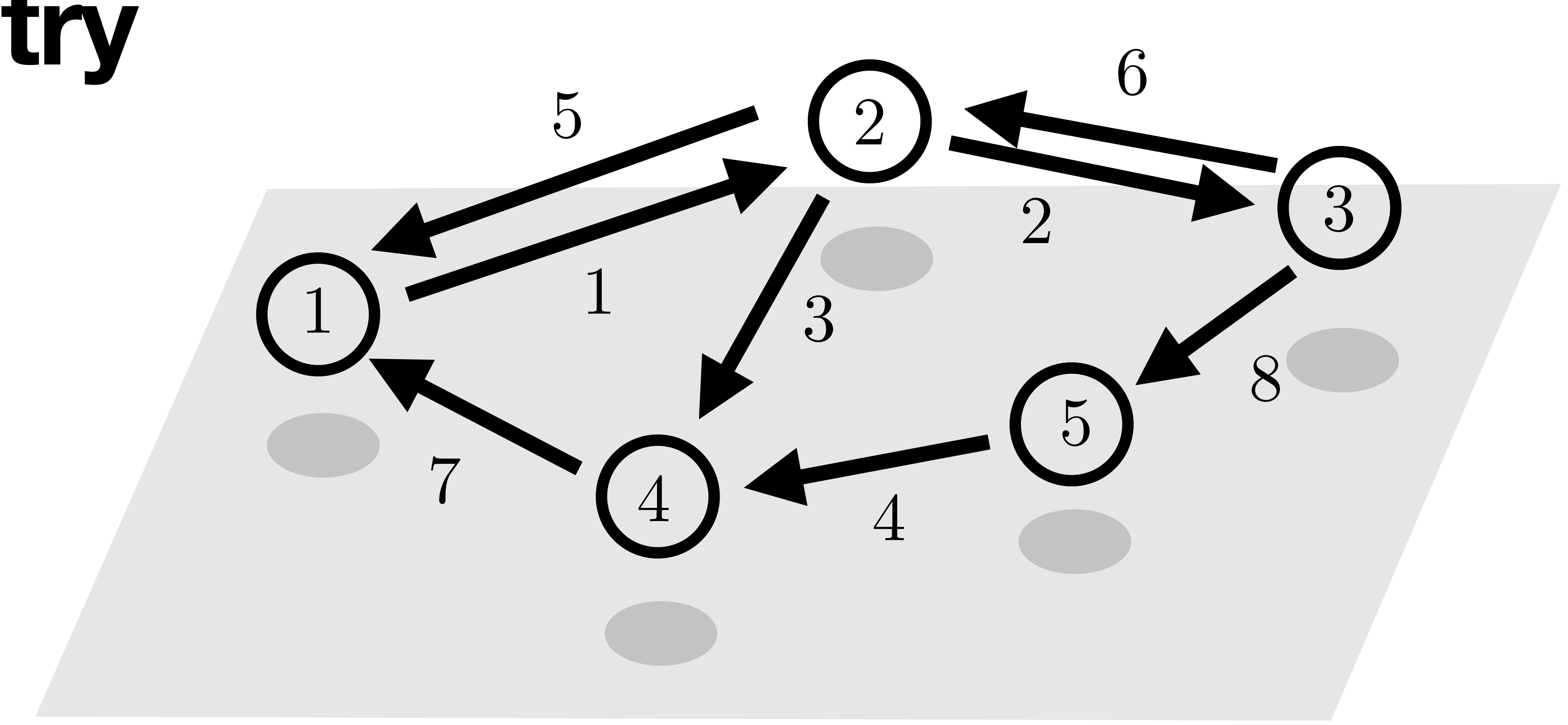
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

...value function on nodes

$$w^T D = \tau^T$$

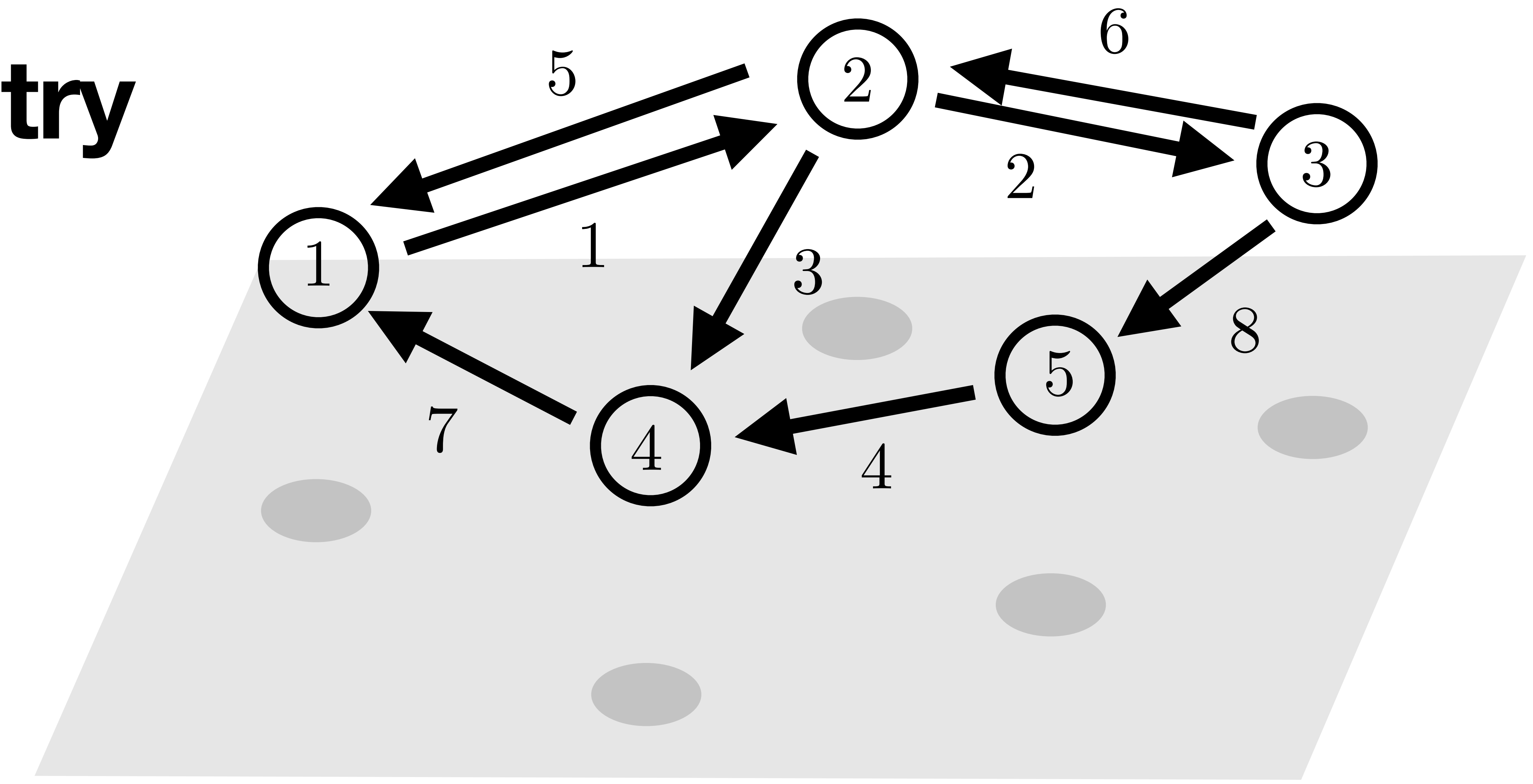
Value function

Edge tension

$$(w^T + \mathbf{1}^T) D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Column Geometry



Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

...value function on nodes

$$w^T D = \tau^T$$

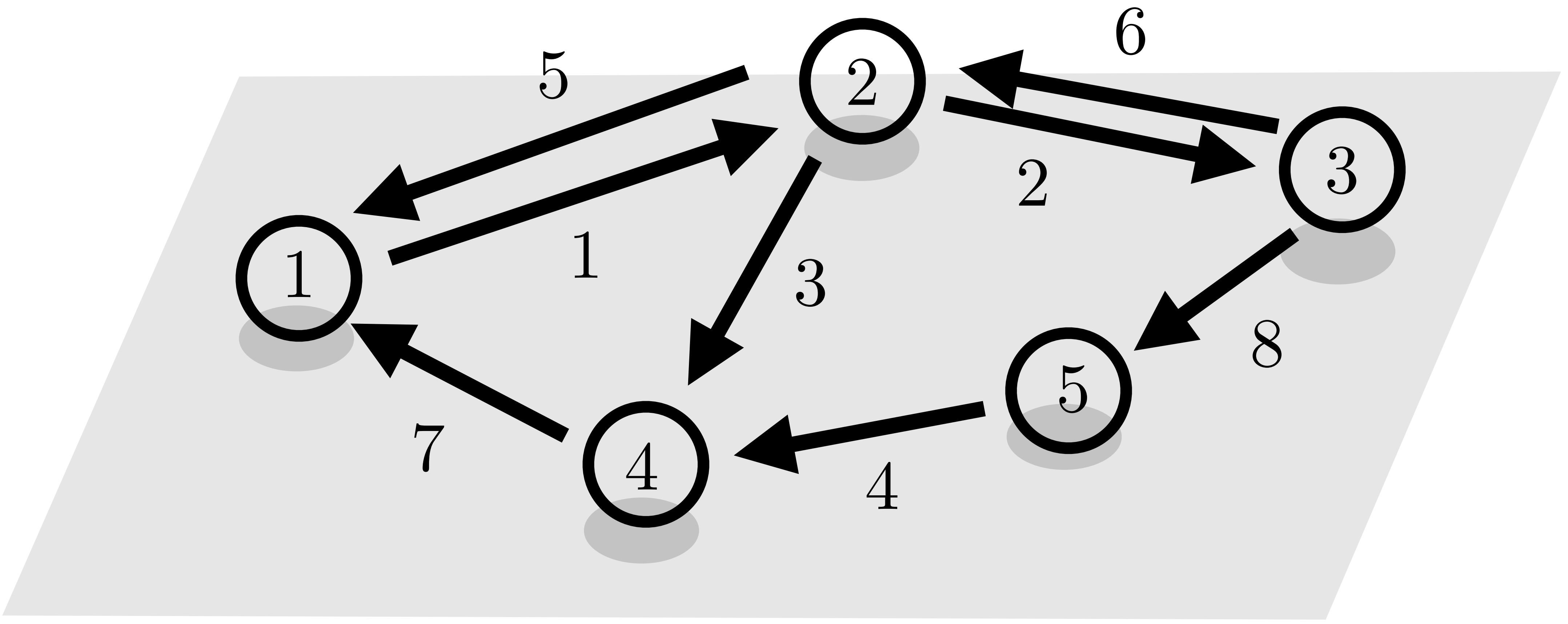
Value function

Edge tension

$$(w^T + \mathbf{1}^T) D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Column Geometry



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ **...mass flow on edges**

Non-conserved flow

$$S = Dx$$

Edge flow vector

$$x = \bar{x} + Cz$$

Specific Solution

Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ **...source-sink on nodes**

...value function on nodes

$$w^T D = \tau^T$$

Value function

Edge tension

$$(w^T + \mathbf{1}^T) D = \tau^T$$

Constant shift (doesn't change tension)

Incidence Matrix - Column Geometry

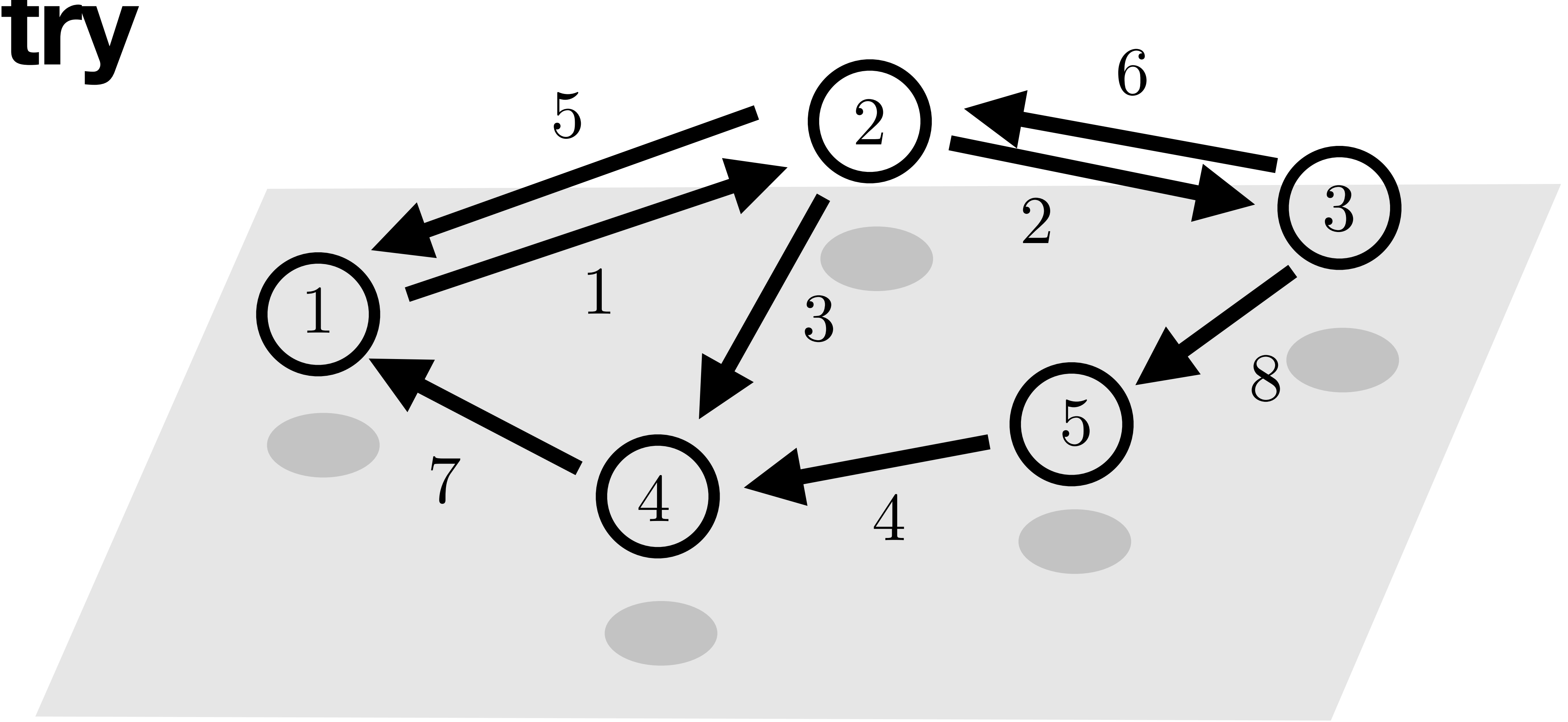
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Domain & Co-Domain Interpretation

Domain: $x \in \mathbb{R}^{|\mathcal{E}|}$ ...mass flow on edges

Non-conserved flow $\quad$ $$S = Dx$$ $\quad$ **Edge flow vector**

$$x = \boxed{\bar{x} + Cz}$$

Specific Solution $\quad$ Cyclic Flow

Co-domain: $w \in \mathbb{R}^{|\mathcal{V}|}$ ...source-sink on nodes
...value function on nodes

$$w^T D = \tau^T$$

Value function $\quad$ **Edge tension**

$$(w^T + \mathbf{1}^T)D = \tau^T$$

Constant shift (doesn't change tension)

Graph Laplacians

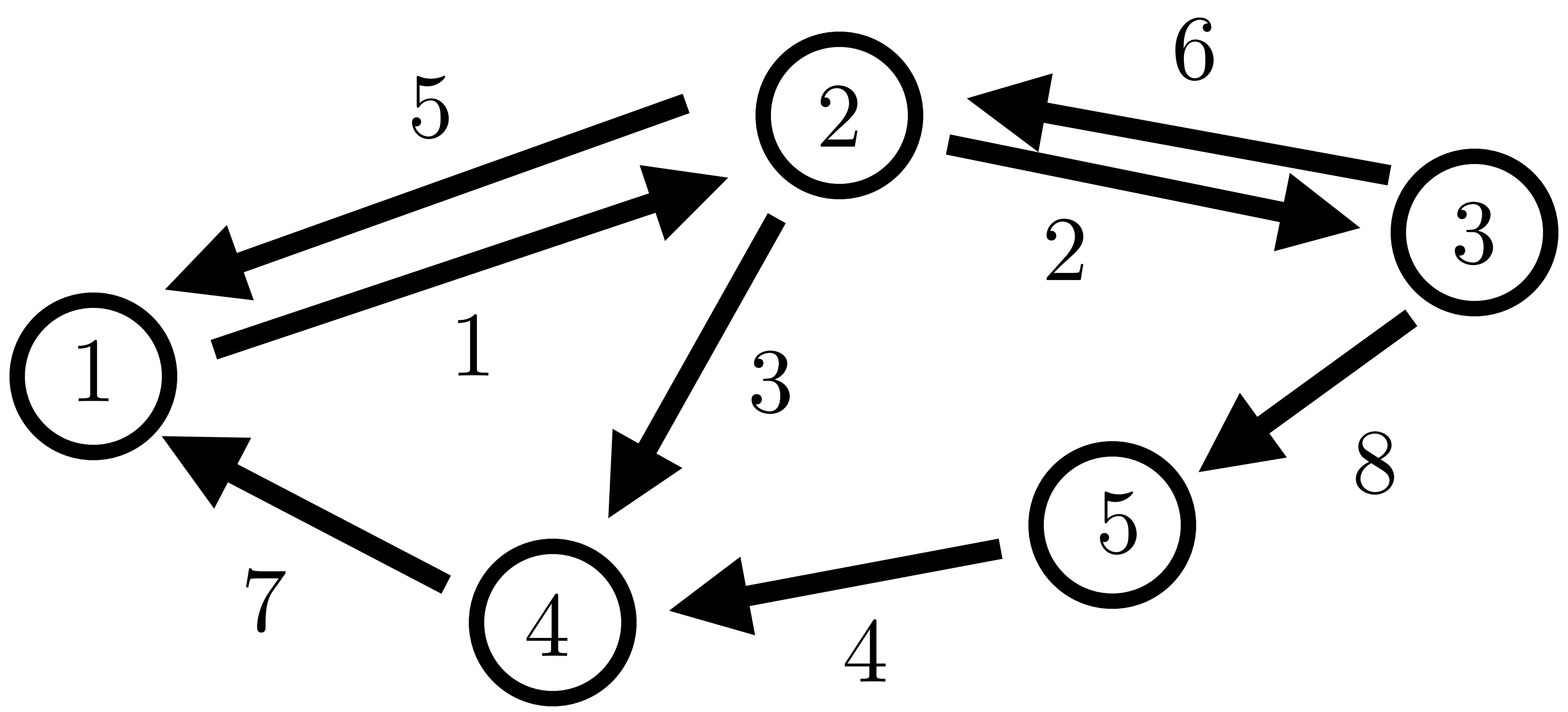
Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

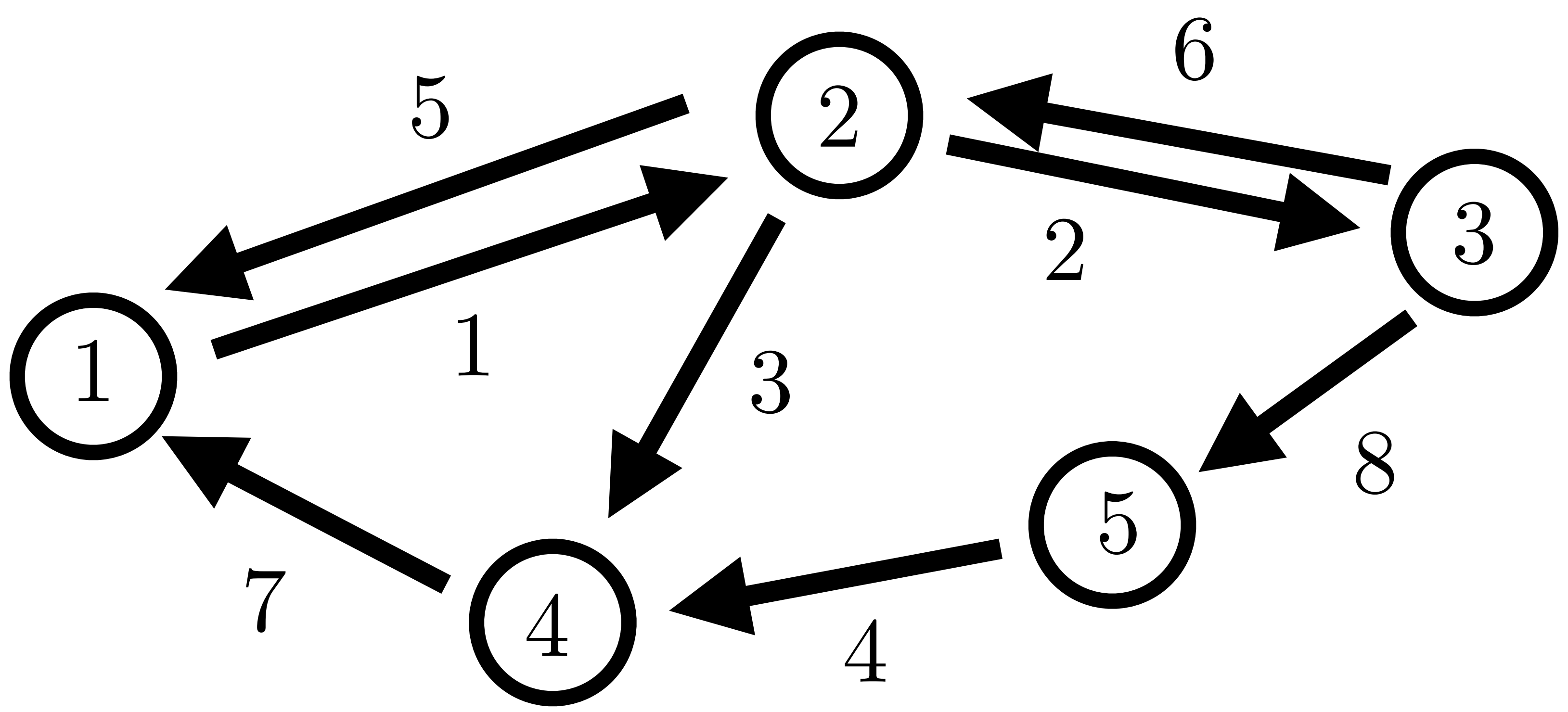
Graph Laplacians

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…** $$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

Inner products of columns

“Relative geometry of columns”

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…** $$A A^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

Inner products of rows

“Relative geometry of rows”

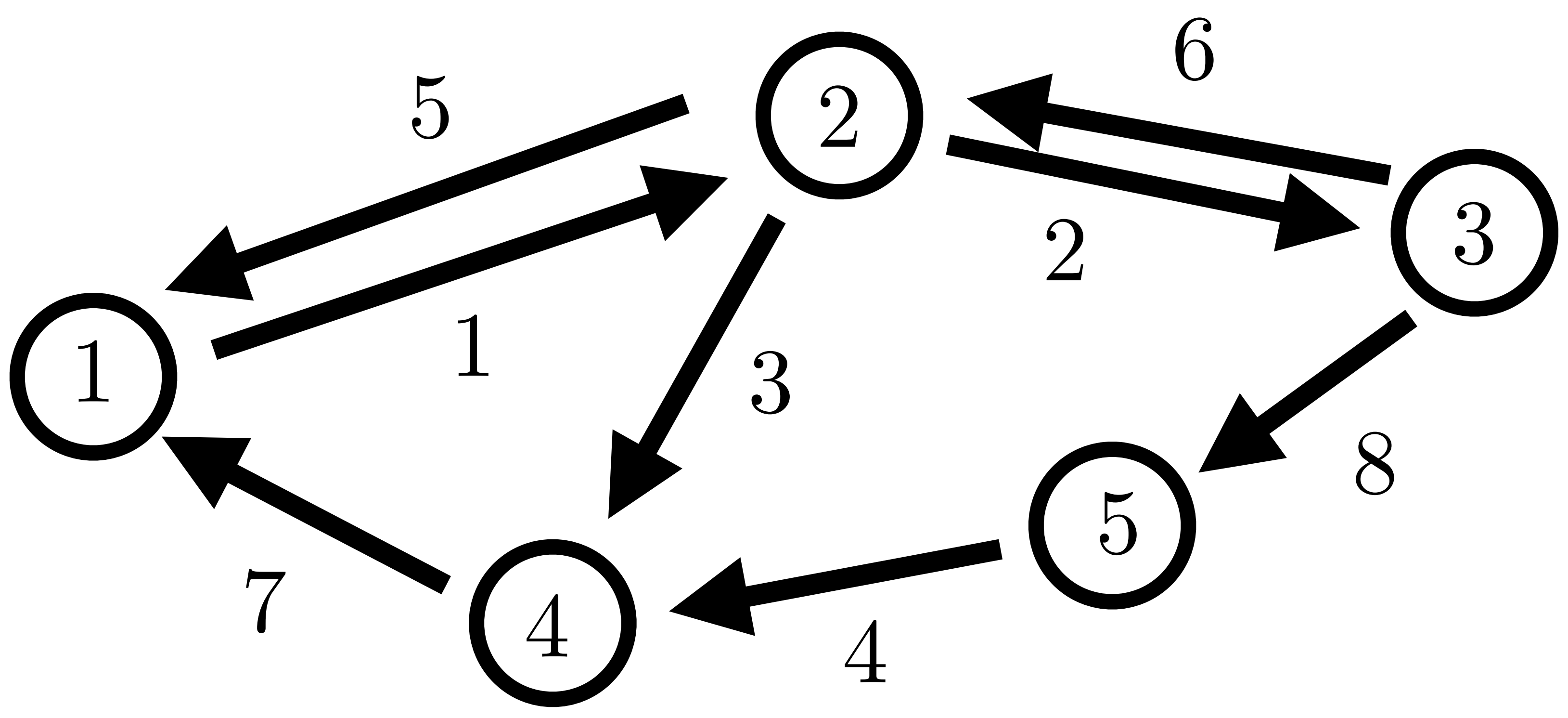
Graph Laplacians

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Review: Shape Matrices

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...** $$A^T A = \begin{bmatrix} A_1^T A_1 & \cdots & A_1^T A_n \\ \vdots & & \vdots \\ A_n^T A_1 & \cdots & A_n^T A_n \end{bmatrix}$$

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...** $$AA^T = \begin{bmatrix} a_1^T a_1 & \cdots & a_1^T a_m \\ \vdots & & \vdots \\ a_m^T a_1 & \cdots & a_m^T a_m \end{bmatrix}$$

RA rotate columns of A…

….relative geometry stays the same.

$$(RA)^T(RA) = A^T R^T R A = A^T A$$

AR rotate rows of A…

….relative geometry stays the same.

$$(AR)(AR)^T = ARR^T A^T = AA^T$$

Graph Laplacians

Graph:

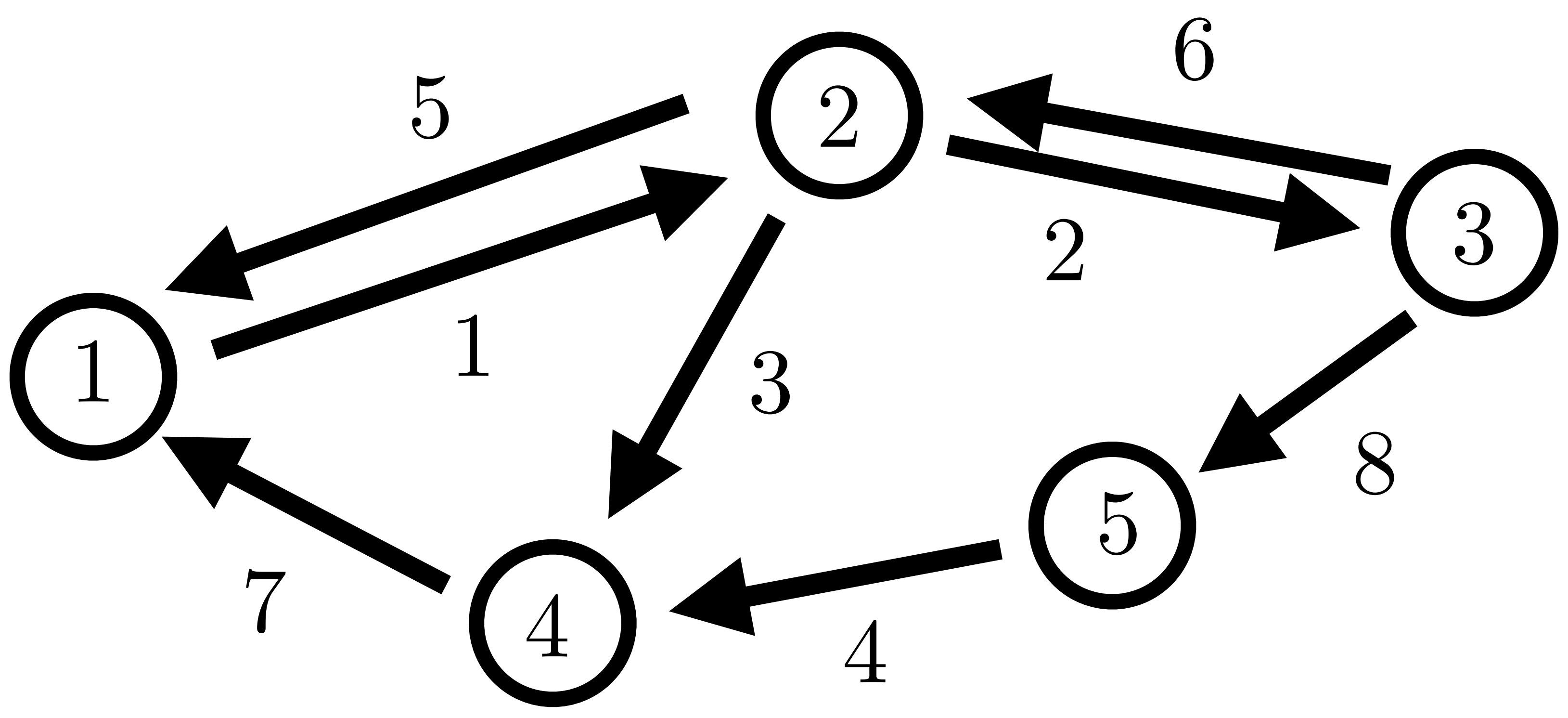
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Review: Shape Matrices

$A^T A$ “Shape” of the columns of A

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Graph Laplacians

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

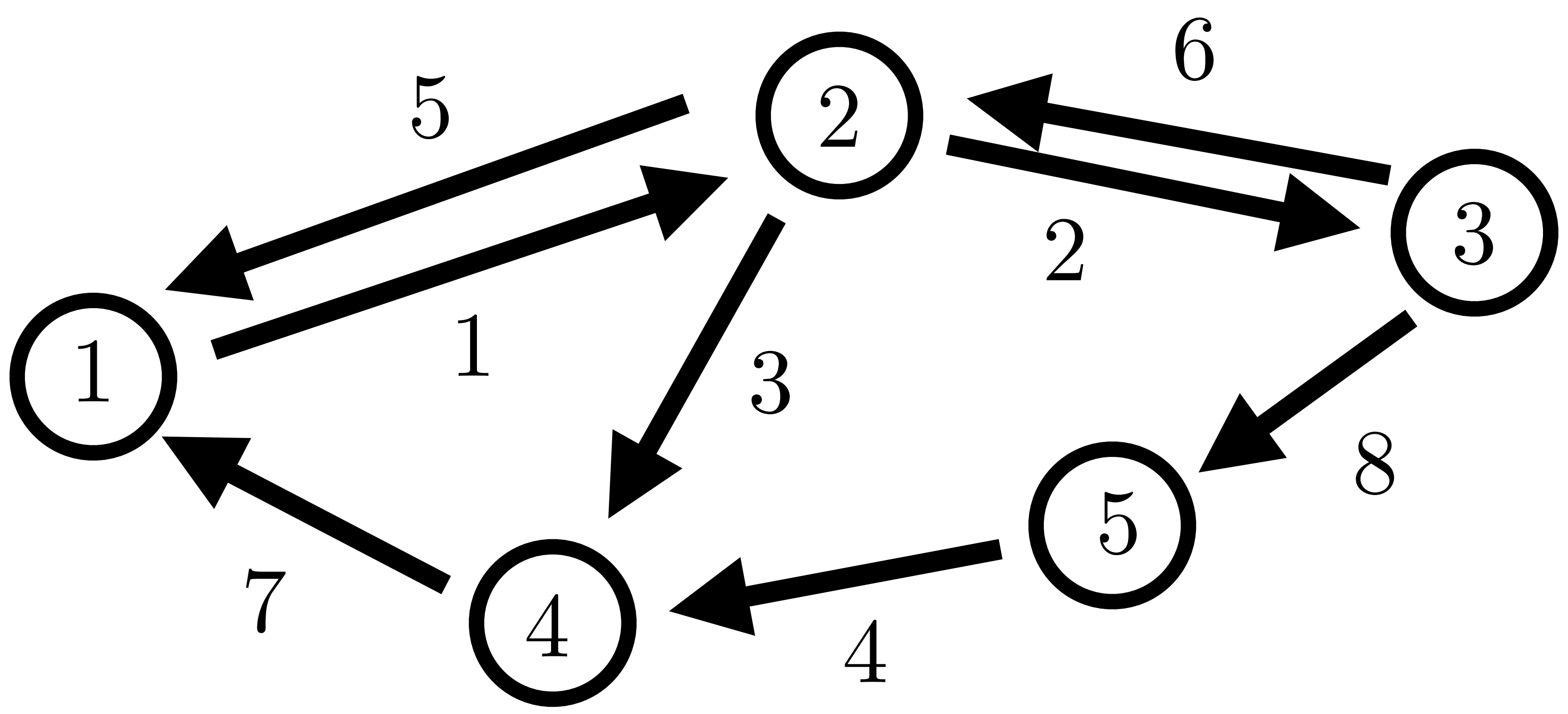
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**



Review: Shape Matrices

$A^T A$ ~~“Shape” of the columns of A~~

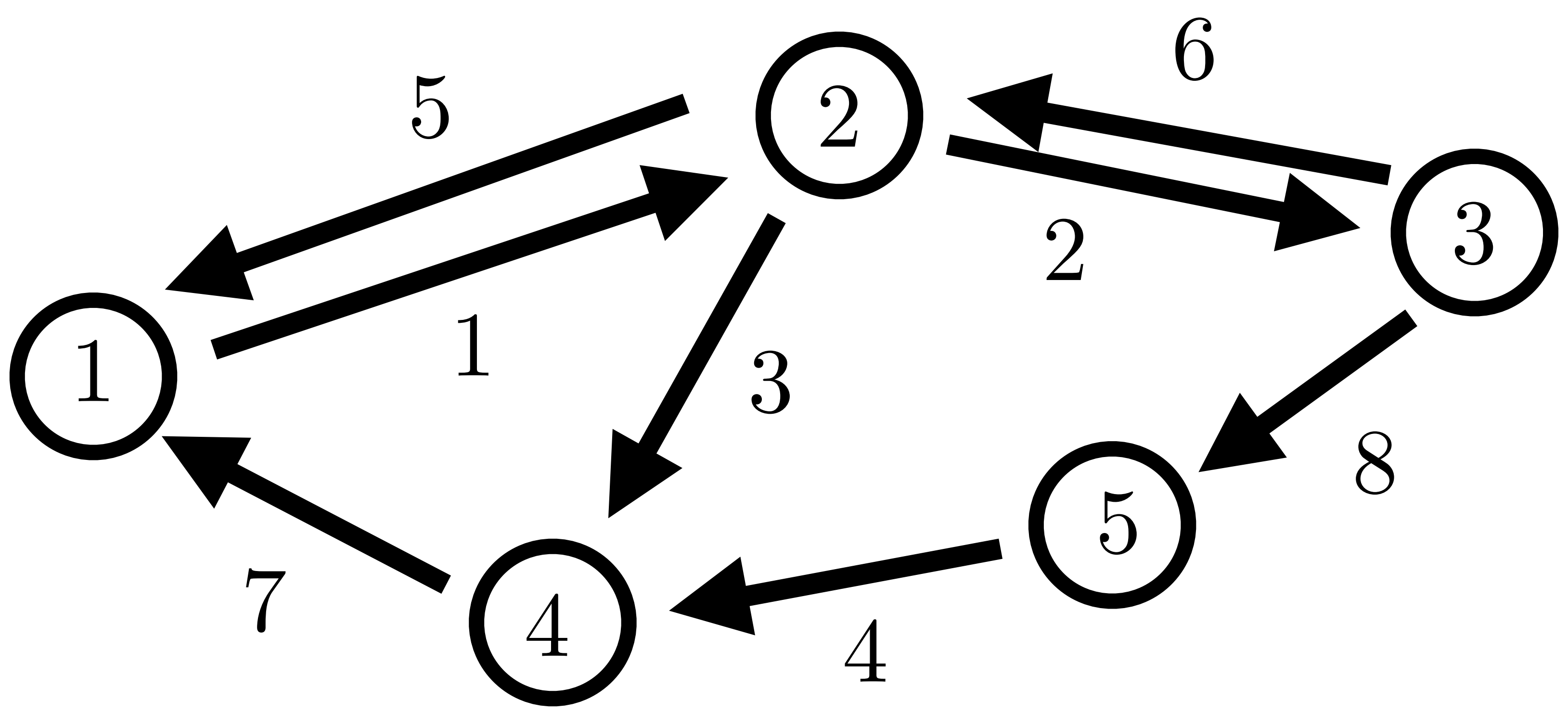
Graph Laplacians

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of the columns of A

More Accurate

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $e = (v, v')$

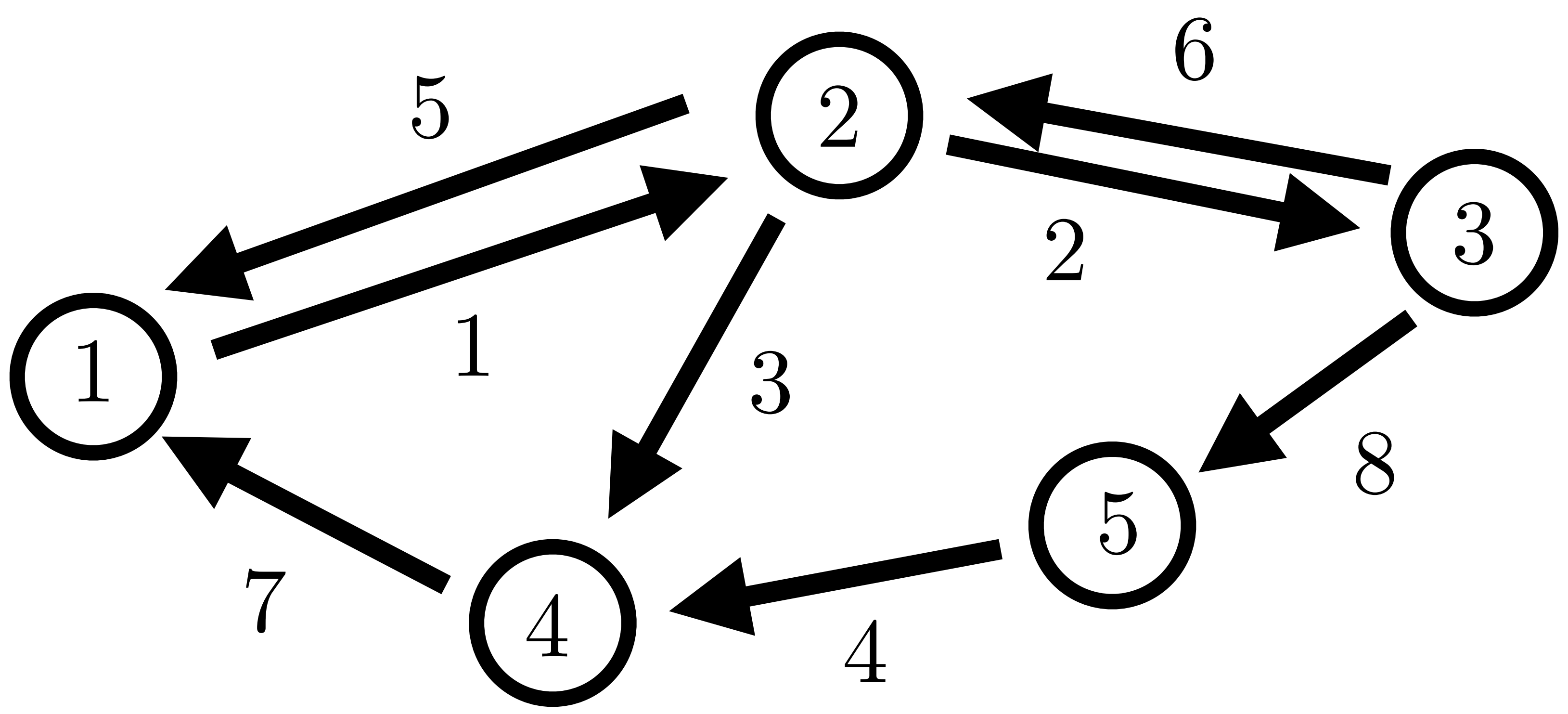
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**



Review: Shape Matrices

$(A^T A)^{1/2}$ “Shape” of columns $(AA^T)^{1/2}$ “Shape” of rows

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

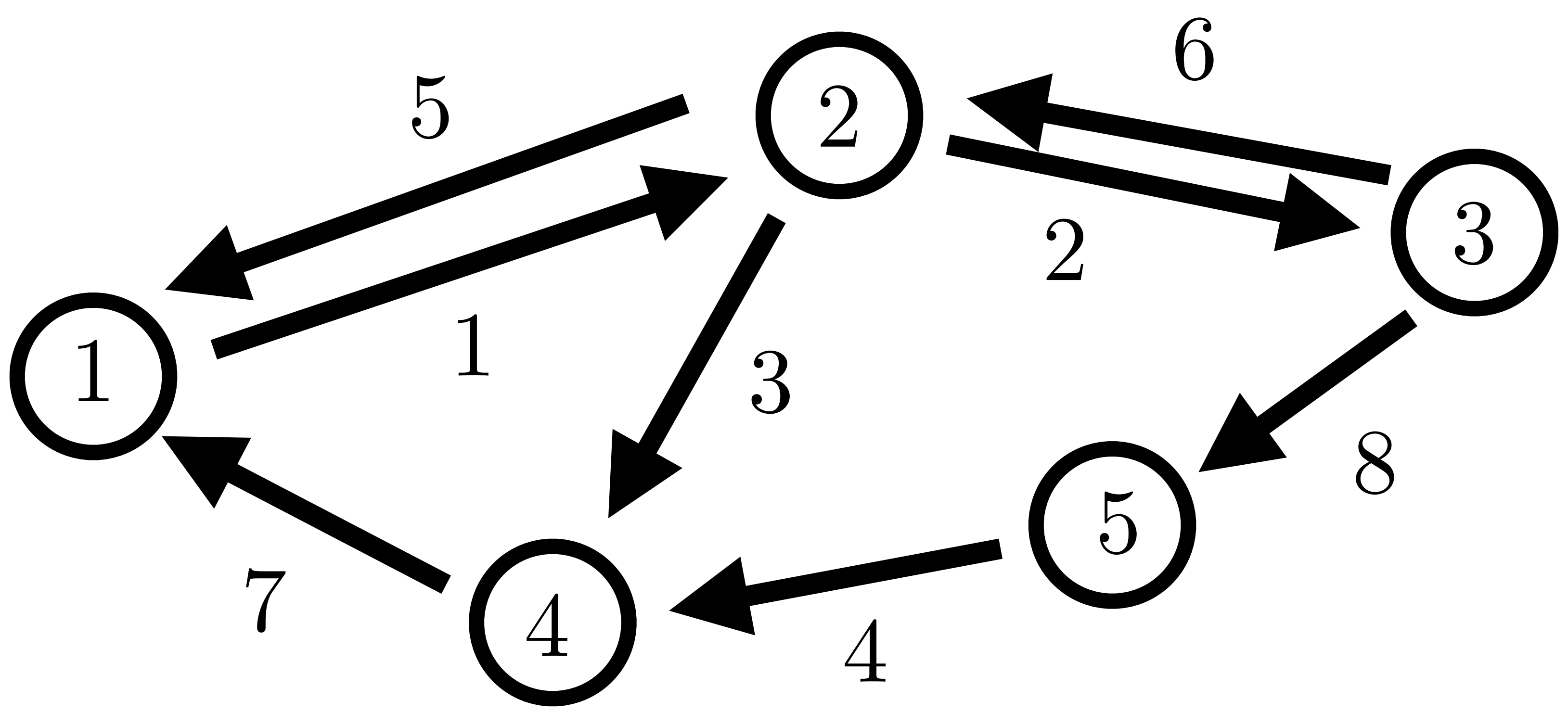
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**



Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

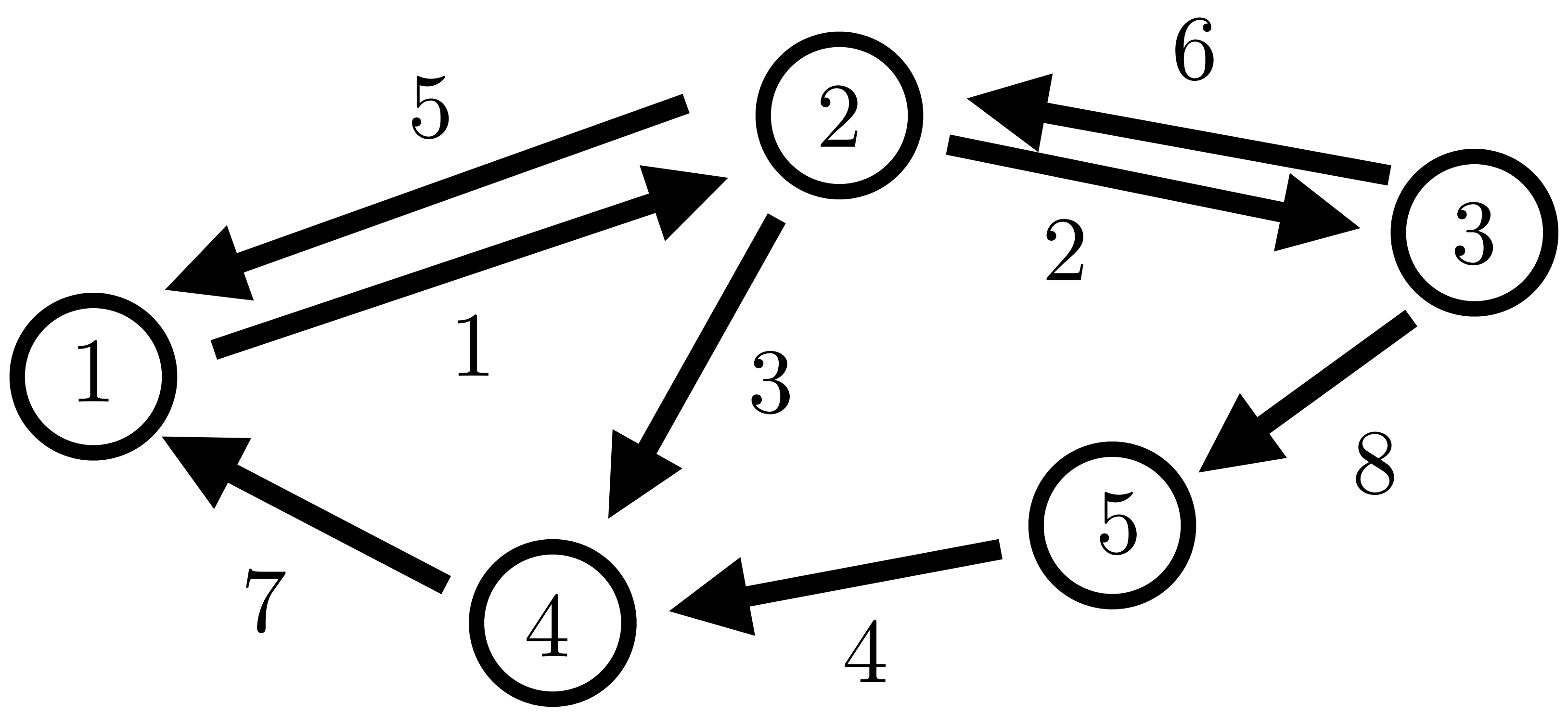
Analogy: $z \in \mathbb{C}$ $\quad |z| = \sqrt{z^* z}$ $\quad z = |z| e^{i\phi}$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^T A)^{1/2}$ “Shape” of columns $\quad (AA^T)^{1/2}$ “Shape” of rows

Analogy: $z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z| e^{i\phi}$

Polar Decomposition

$$A = \underbrace{A(A^T A)^{-1/2}}_{\text{Rotation}} \cdot \underbrace{(A^T A)^{1/2}}_{\text{PSD “shape"}}$$

“Column version”

Graph Laplacians

Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

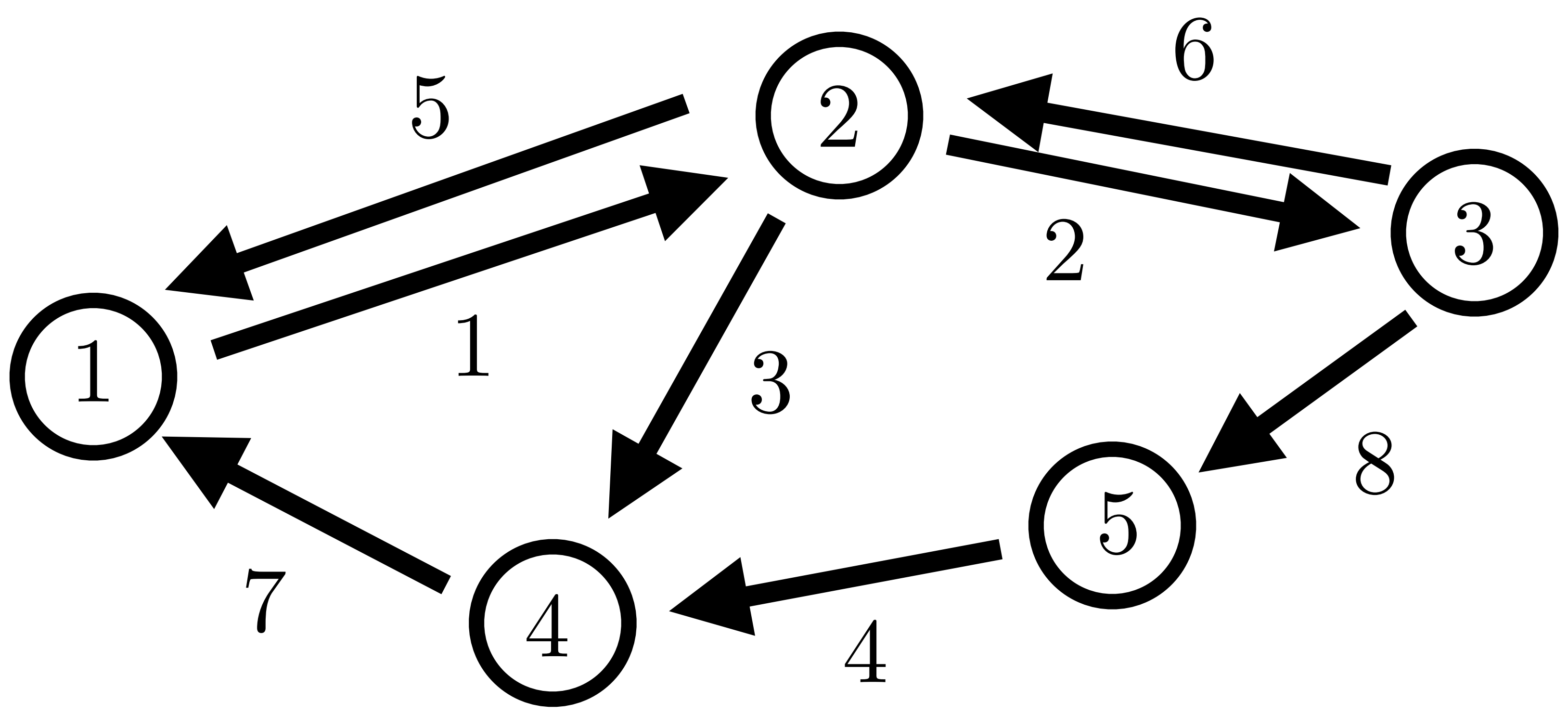
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**



Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $(AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C}$ $|z| = \sqrt{z^* z}$ $z = |z| e^{i\phi}$

Polar Decomposition

$$A = A(A^T A)^{-1/2} \cdot (A^T A)^{1/2}$$

Rotation PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" Rotation

"Row version"

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

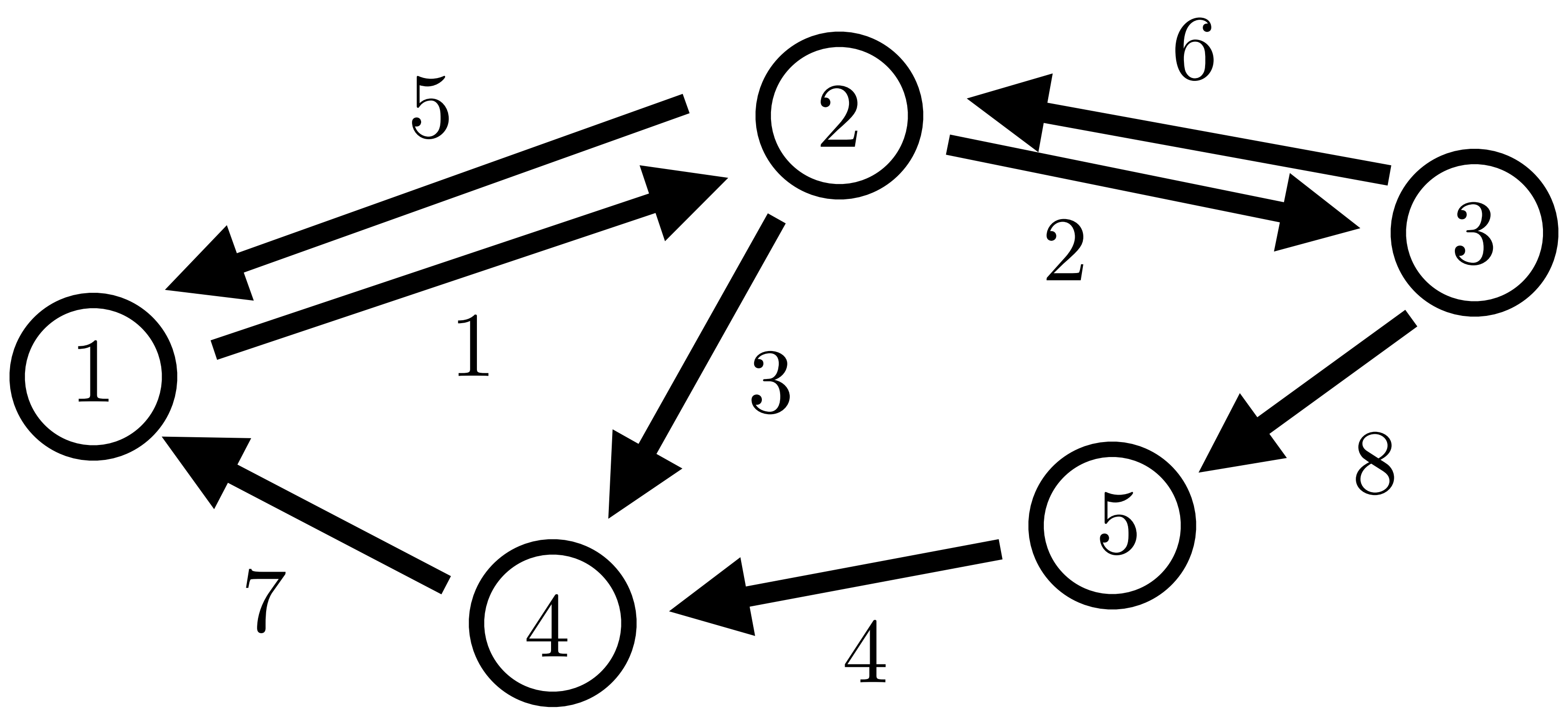
Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**



Review: Shape Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

Analogy: $z \in \mathbb{C} \quad |z| = \sqrt{z^* z} \quad z = |z|e^{i\phi}$

Polar Decomposition

$$A = A(A^TA)^{-1/2} \cdot (A^TA)^{1/2}$$

Rotation PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" Rotation

"Row version"

Checking rotation…

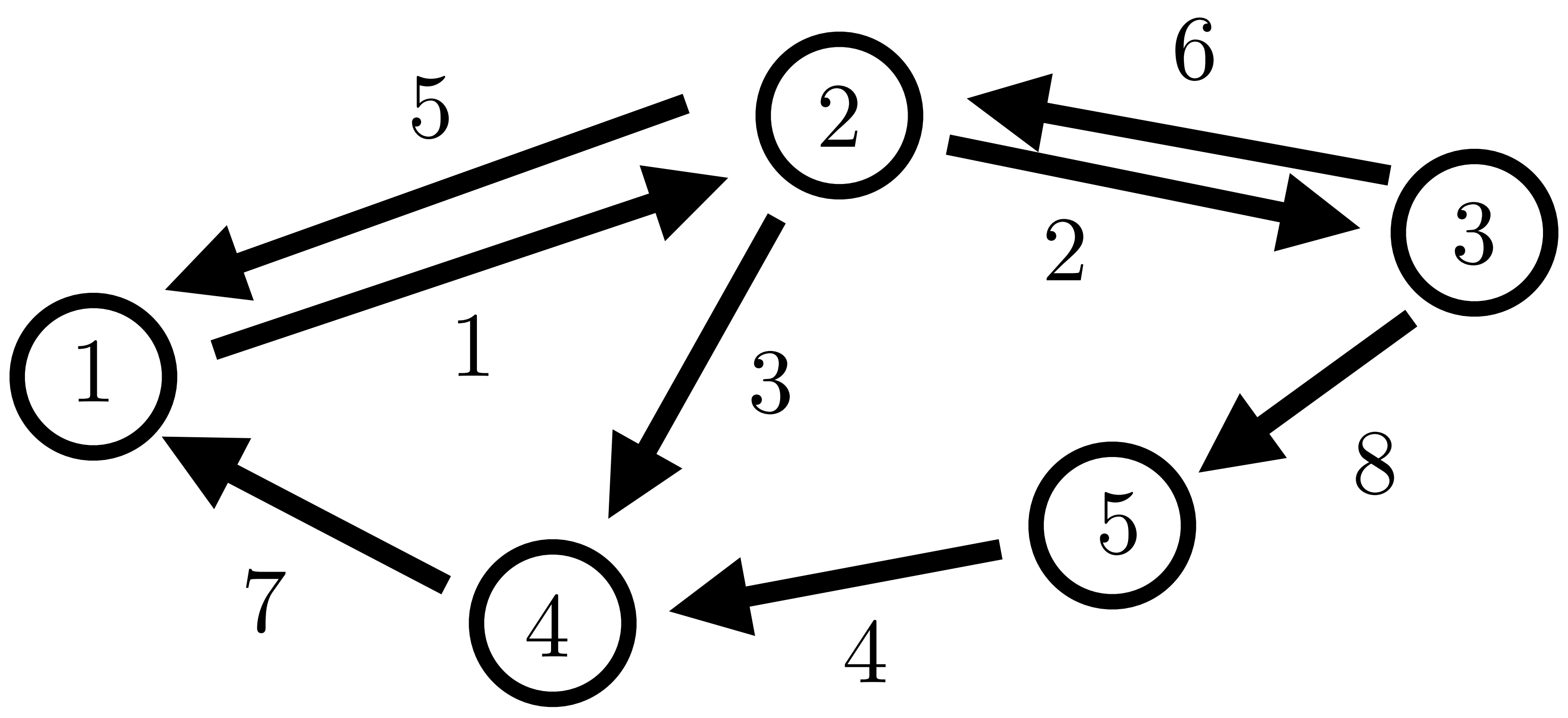
$$(A^TA)^{-1/2} A^TA (A^TA)^{-1/2} = I$$

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^TA)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Polar Decomposition

$$A = A(A^TA)^{-1/2} \cdot (A^TA)^{1/2}$$

Rotation $\quad$ PSD "shape"

"Column version"

$$A = (AA^T)^{1/2} \cdot (AA^T)^{-1/2} A$$

PSD "shape" $\quad$ Rotation

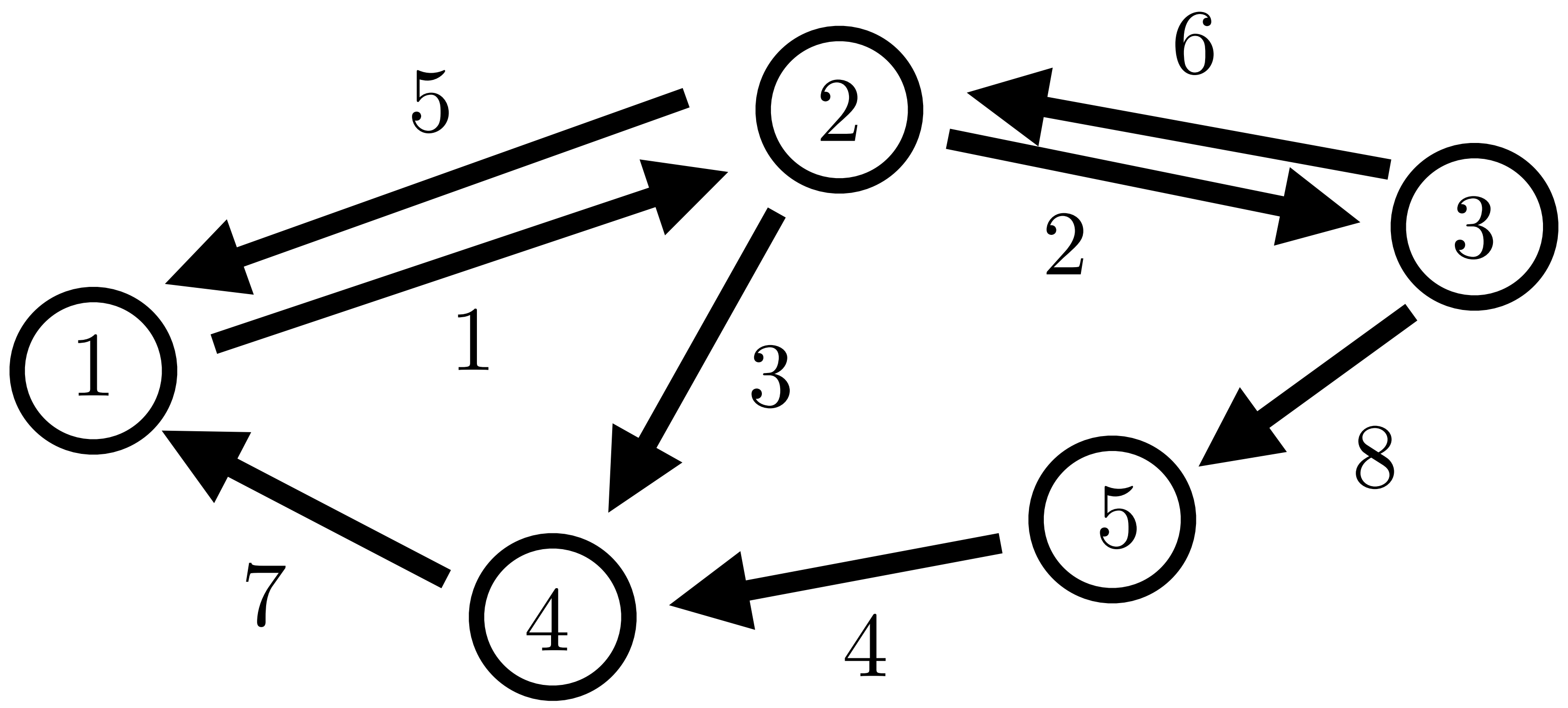
"Row version"

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^TA)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes $\quad (A^TA)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \quad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$

Polar Decomposition

$$A = UV^T \cdot V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$ **"Column version"**

Rotation $\quad$ PSD "shape"

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T \cdot UV^T$$ **"Row version"**

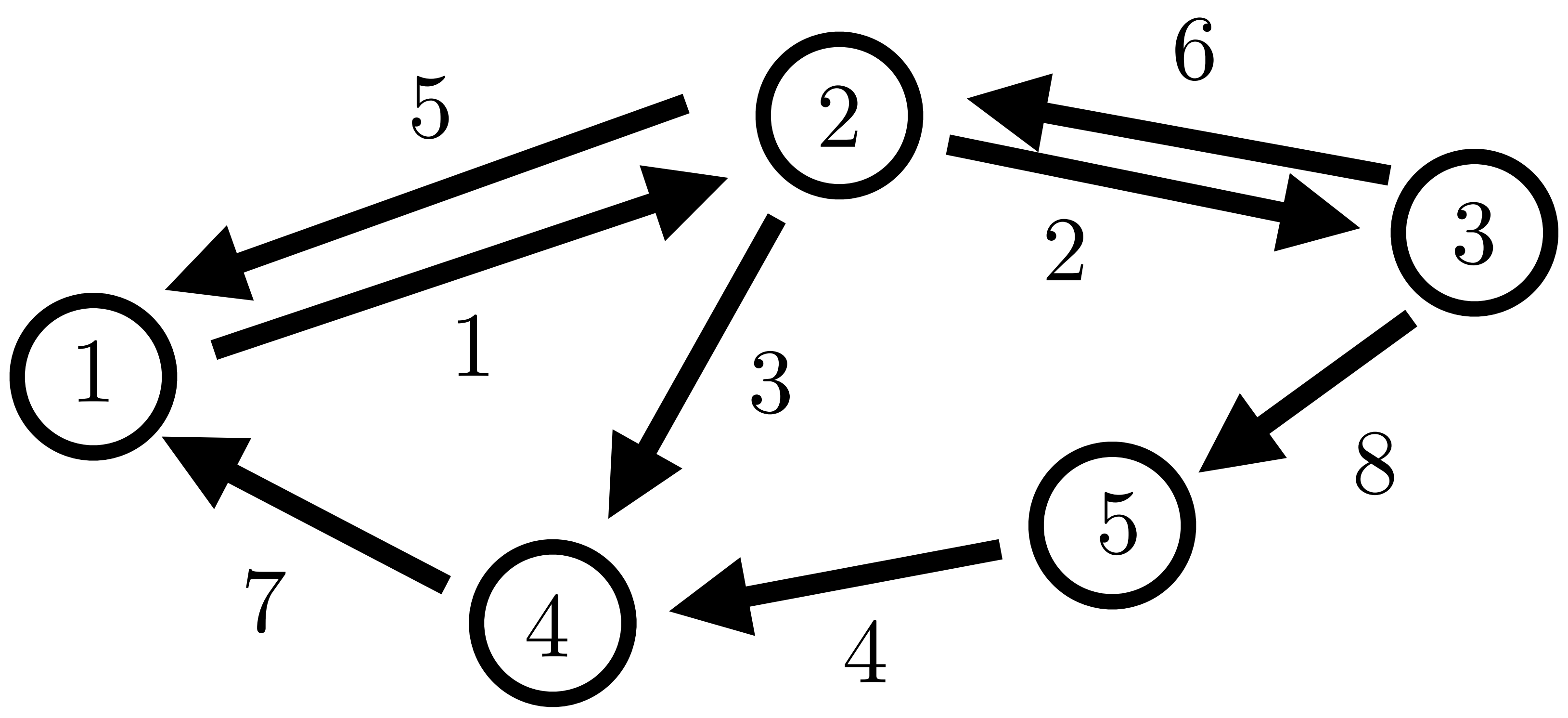
PSD "shape" $\quad$ Rotation

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols…**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows…**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

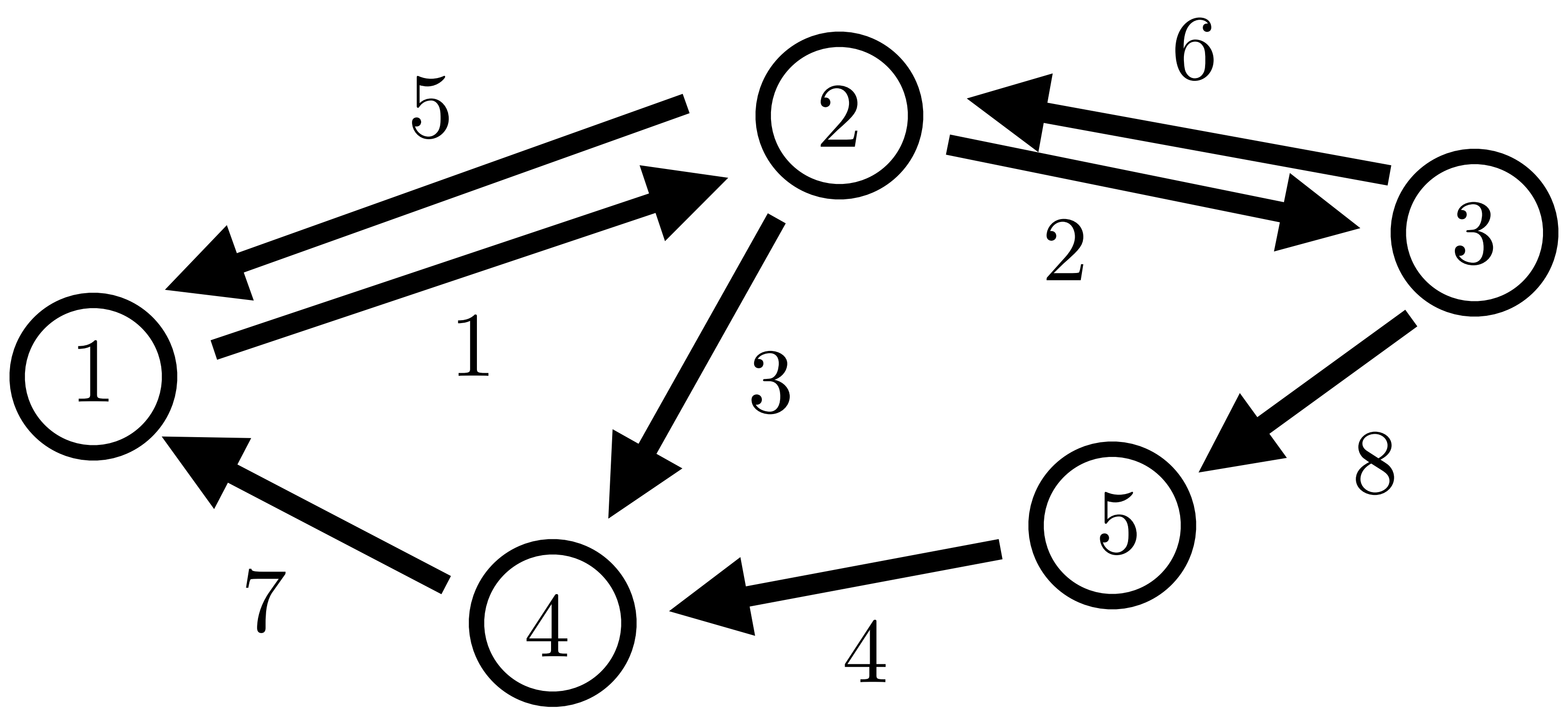
EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Singular Value Decomposition

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

General Matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$ **cols...**

$$= \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$ **rows...**

Review: Shape Matrices

$(A^T A)^{1/2}$ "Shape" of columns $\quad (AA^T)^{1/2}$ "Shape" of rows

EVD of Shapes

$$(A^T A)^{1/2} = V \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T \qquad (AA^T)^{1/2} = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$$

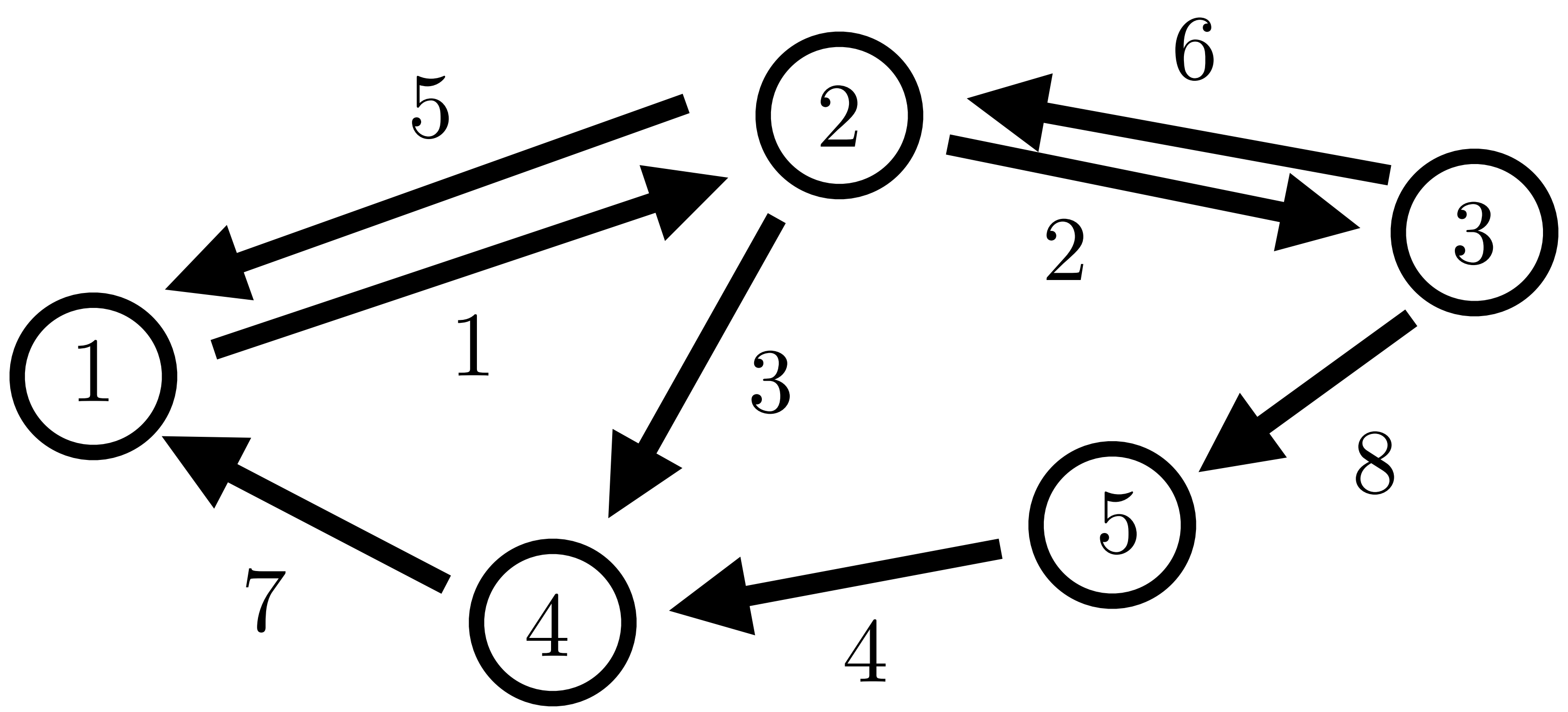
Singular Value Decomposition

$$A = \ U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

$$U' = AV'\Sigma^{-1} \qquad {V'}^T = \Sigma^{-1}{U'}^T A$$

for singular vectors w/ non-zero values

Graph Laplacians



Graph: **Vertices** $v \in \mathcal{V}$

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ **Edges** $e \in \mathcal{E}$ $e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

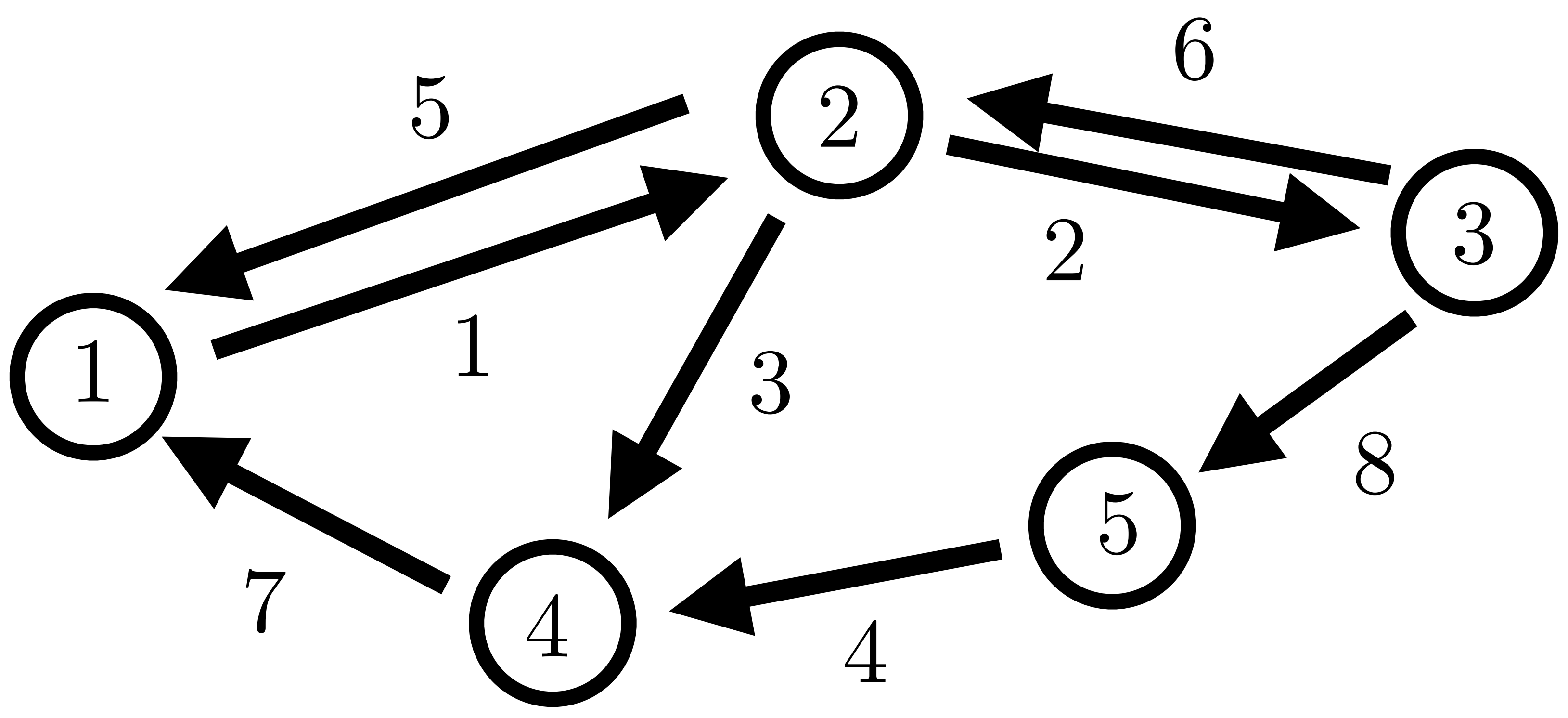
Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Laplacian $L = DD^T$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

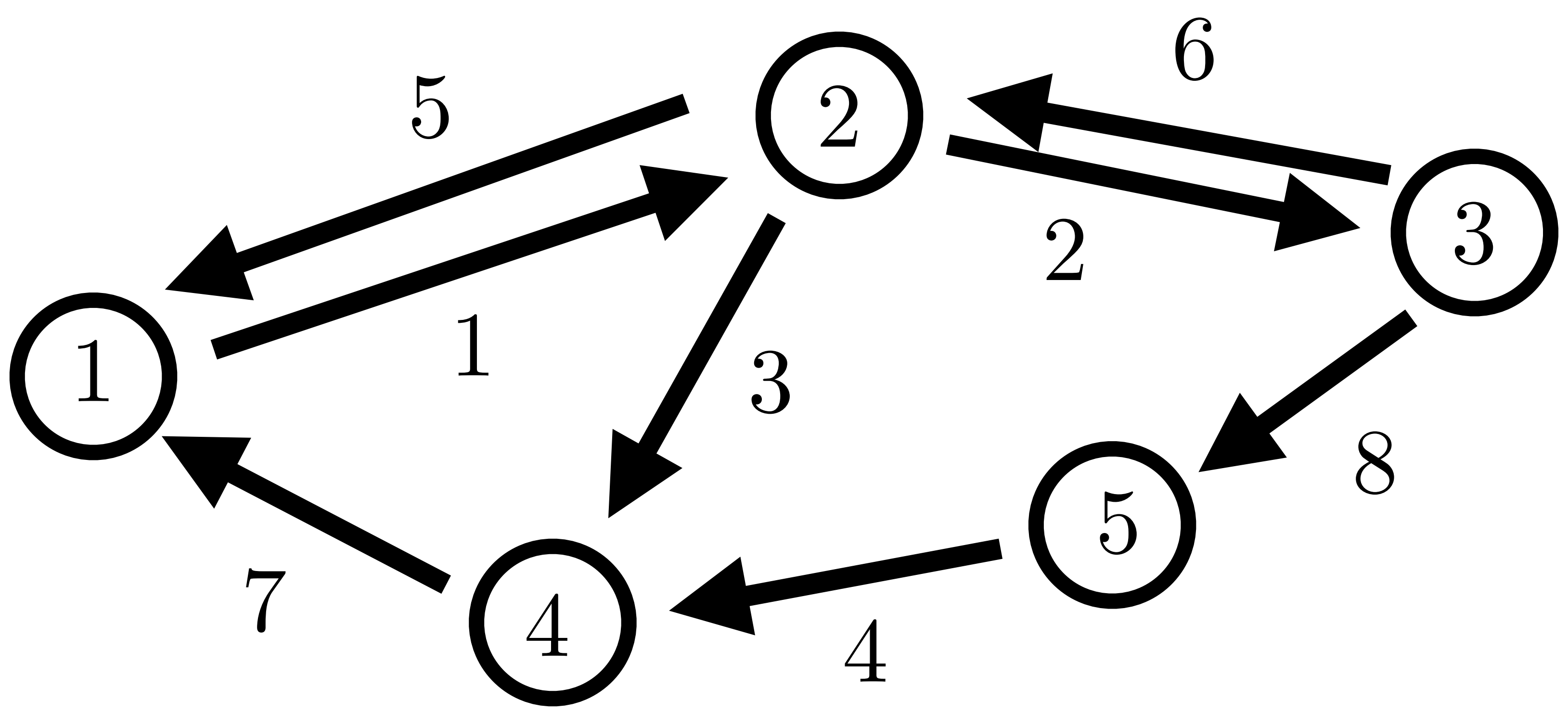
Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

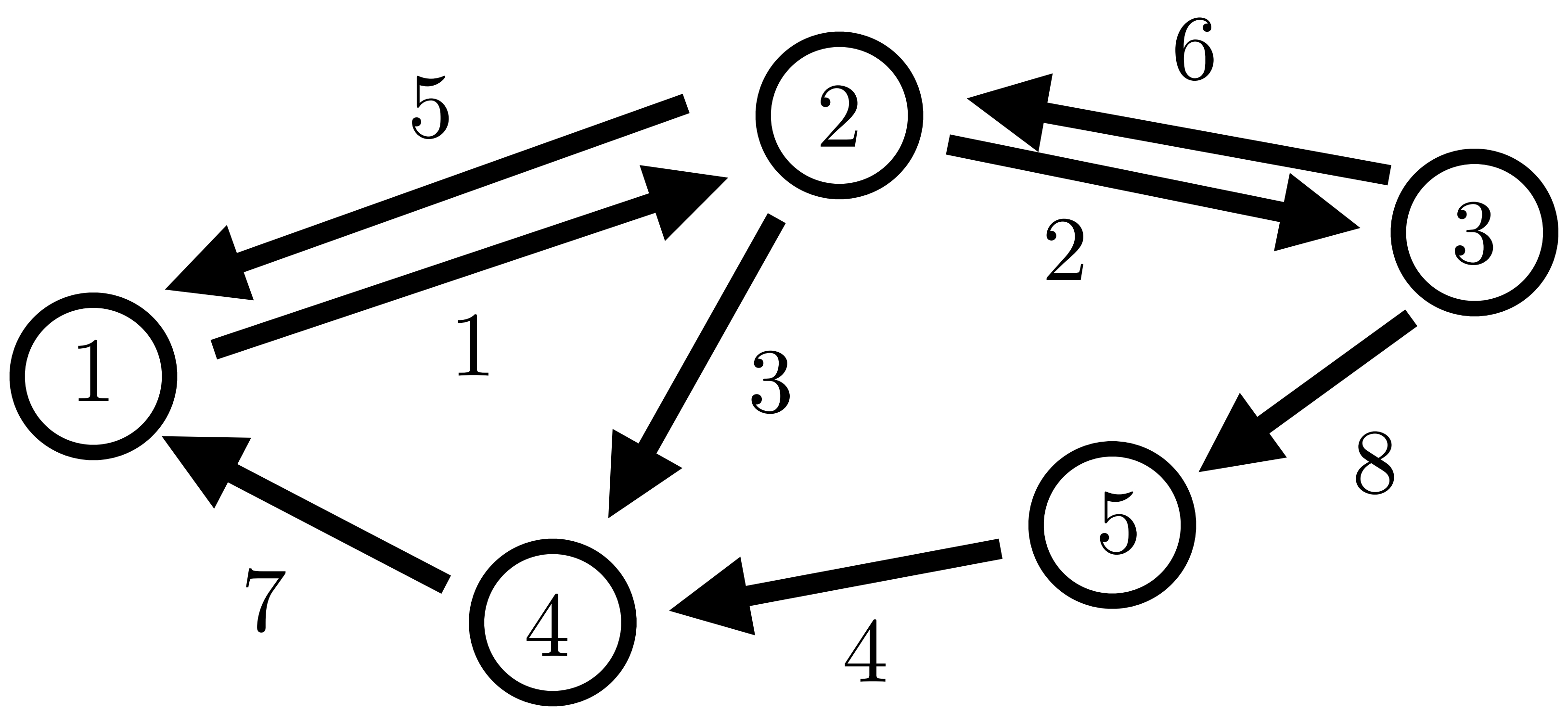
Laplacian $\quad L = DD^T$

Action:

$$Lu = \underbrace{\left[\ D\ \right]\underbrace{\left[\ D^T\ \right]\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{…tension created in edges}}}_{\text{… summed resulting tension on nodes}}$$

"heights" of nodes

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

Action:

$$Lu = \underbrace{\left[\; D \;\right] \underbrace{\left[\; D^T \;\right] \begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges}}}_{\text{... summed resulting tension on nodes}}$$

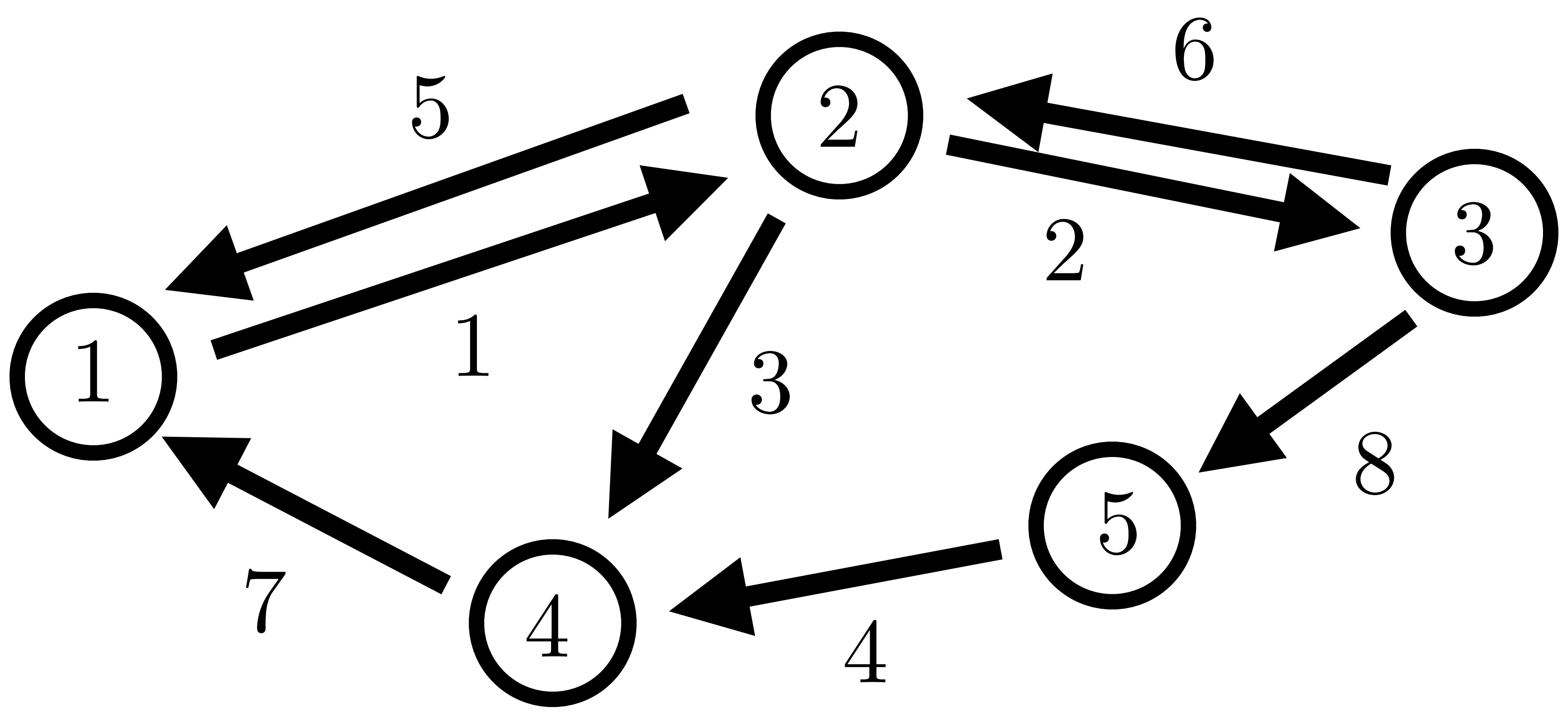
"heights" of nodes

Linear ODE $\quad \dot{u} = -Lu$

Eigenvectors are oscillation modes

"Vibration modes" of a graph

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

$$L = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

$$= \begin{bmatrix} | & | \\ U' & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & U'^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix} \qquad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Laplacian eigenvectors

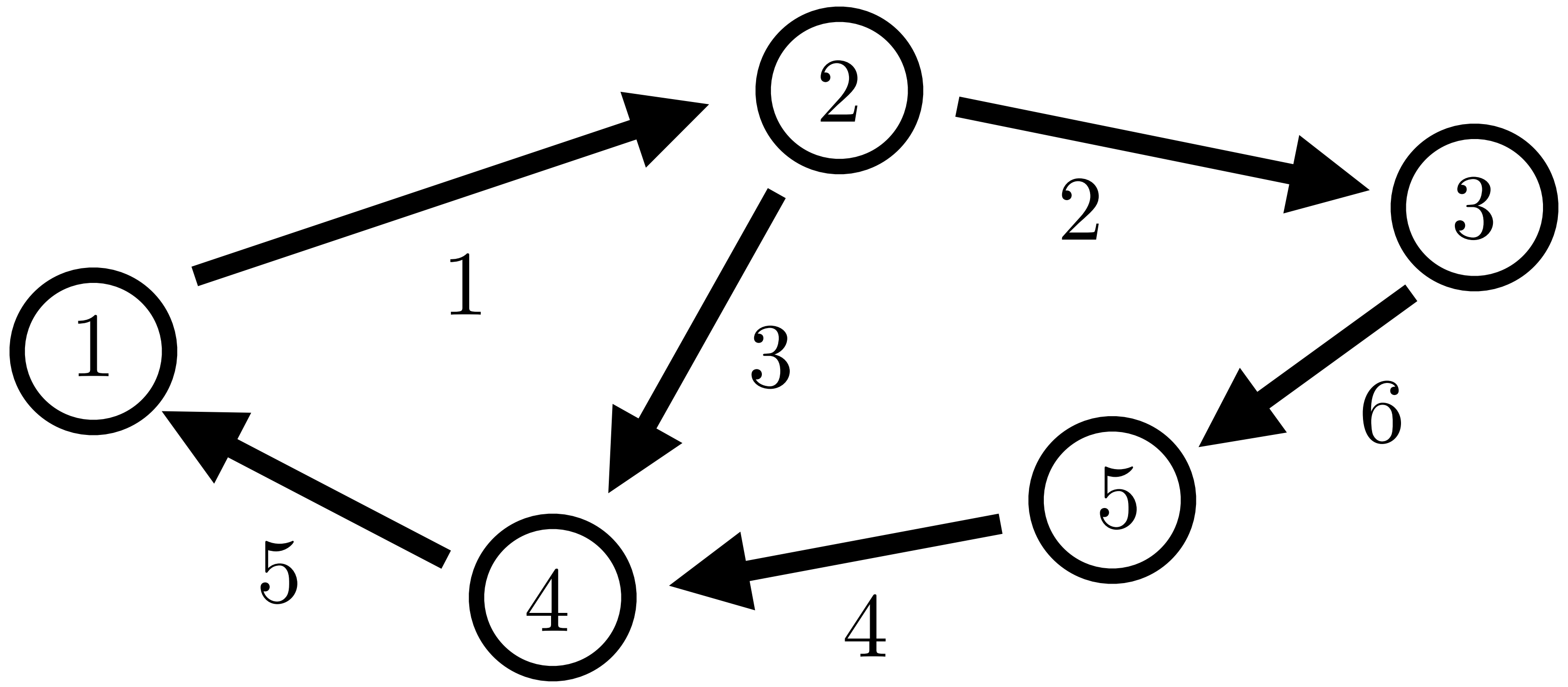
(Incidence matrix left singular vectors)

Graph Laplacians

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T$

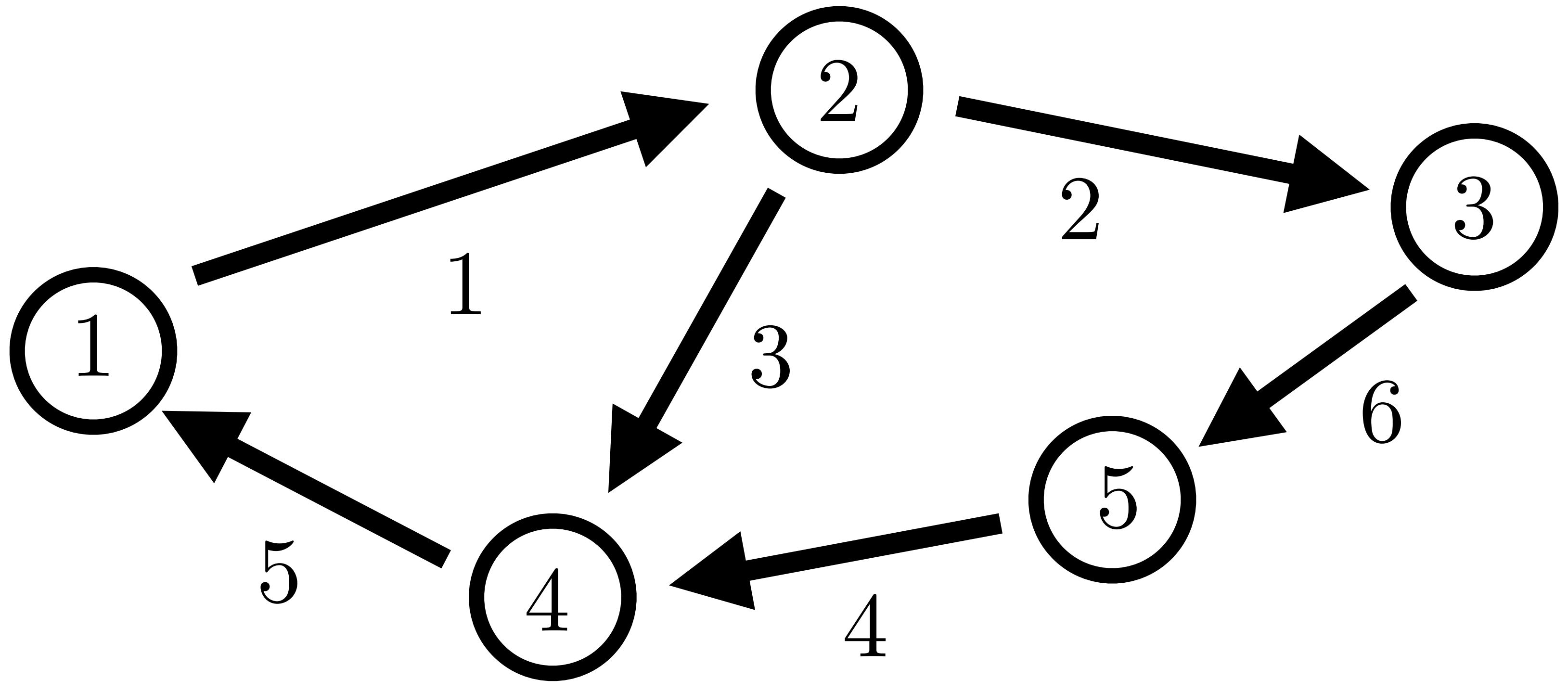
$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Degree & Adjacency Matrix

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

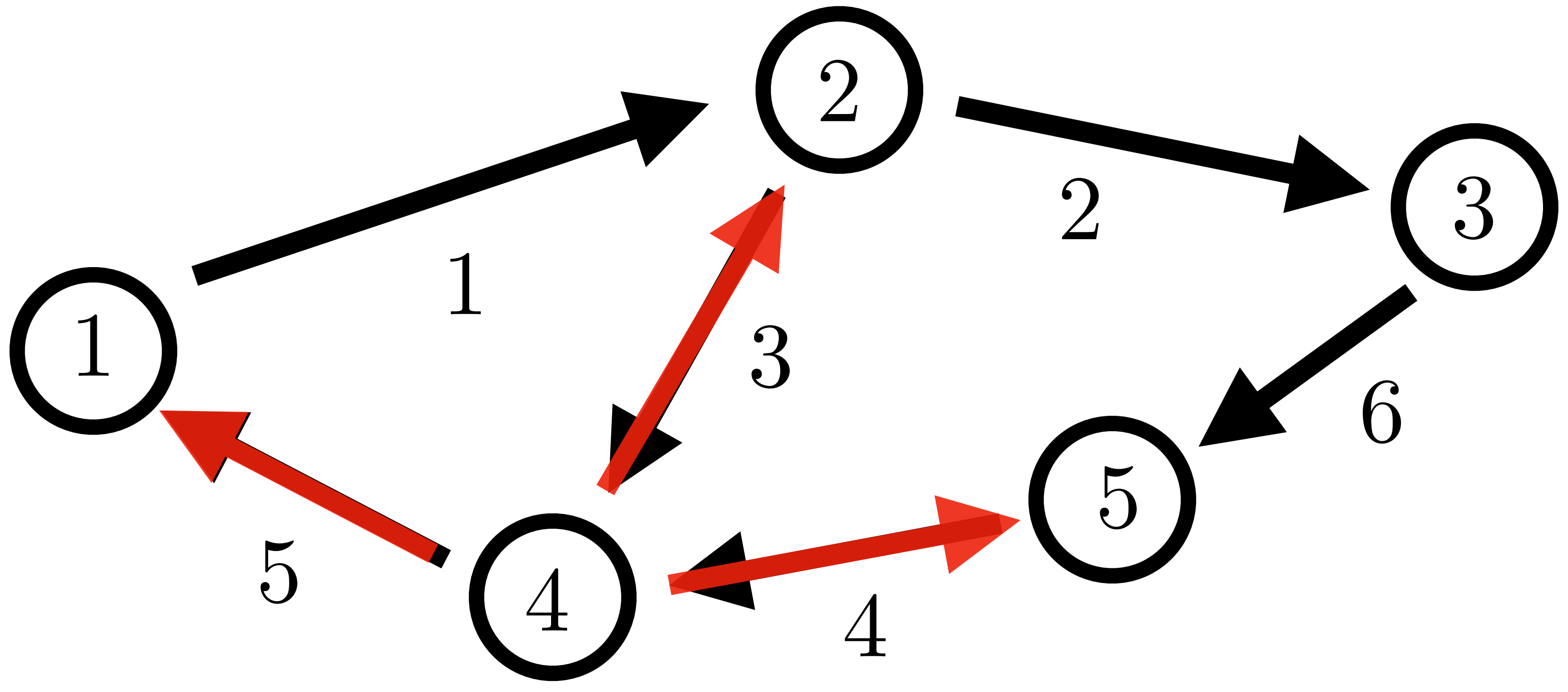
Laplacian $L = DD^T$ Independent of edge direction

$$L = \Delta - A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Degree Matrix $\left[\ \Delta\ \right]_{vv} = |\mathcal{N}_v|$ **diagonal**

Adjacency Matrix $\left[\ A\ \right]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

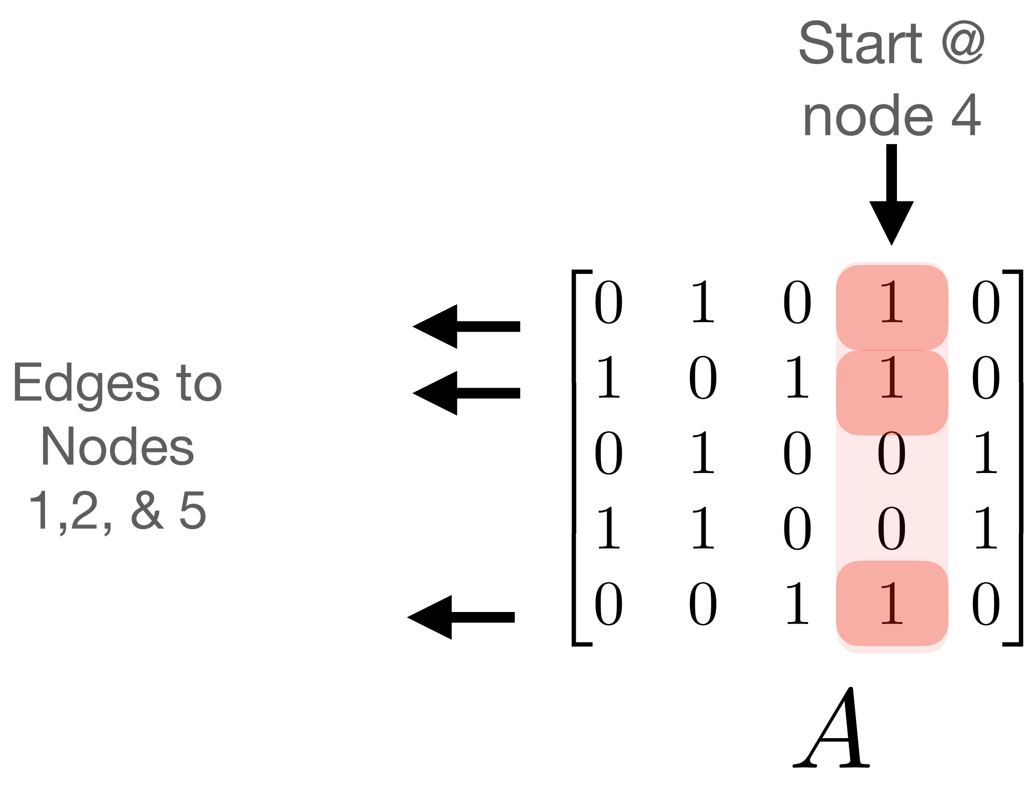
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Adjacency Matrix



Start @ node 4

Edges to Nodes 1,2, & 5

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A

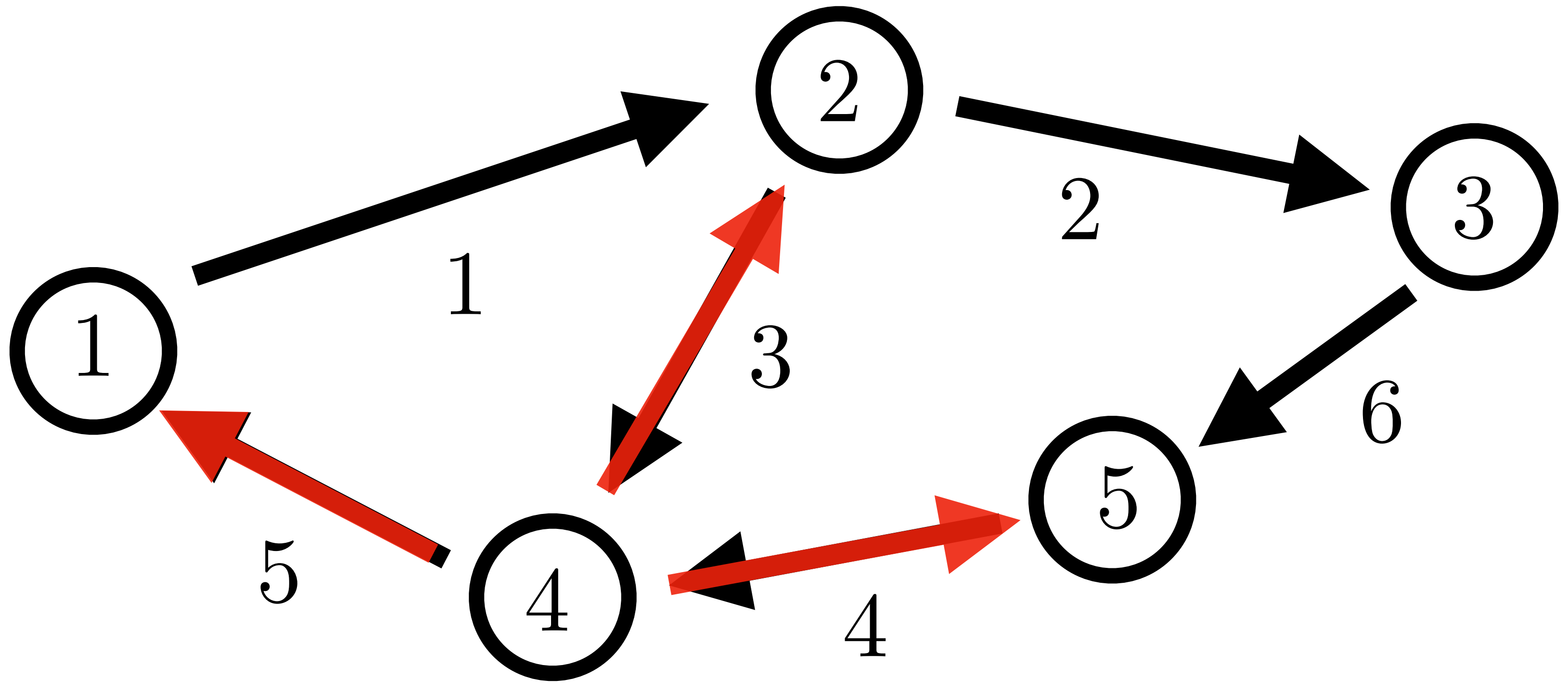
Adjacency Matrix

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$



Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

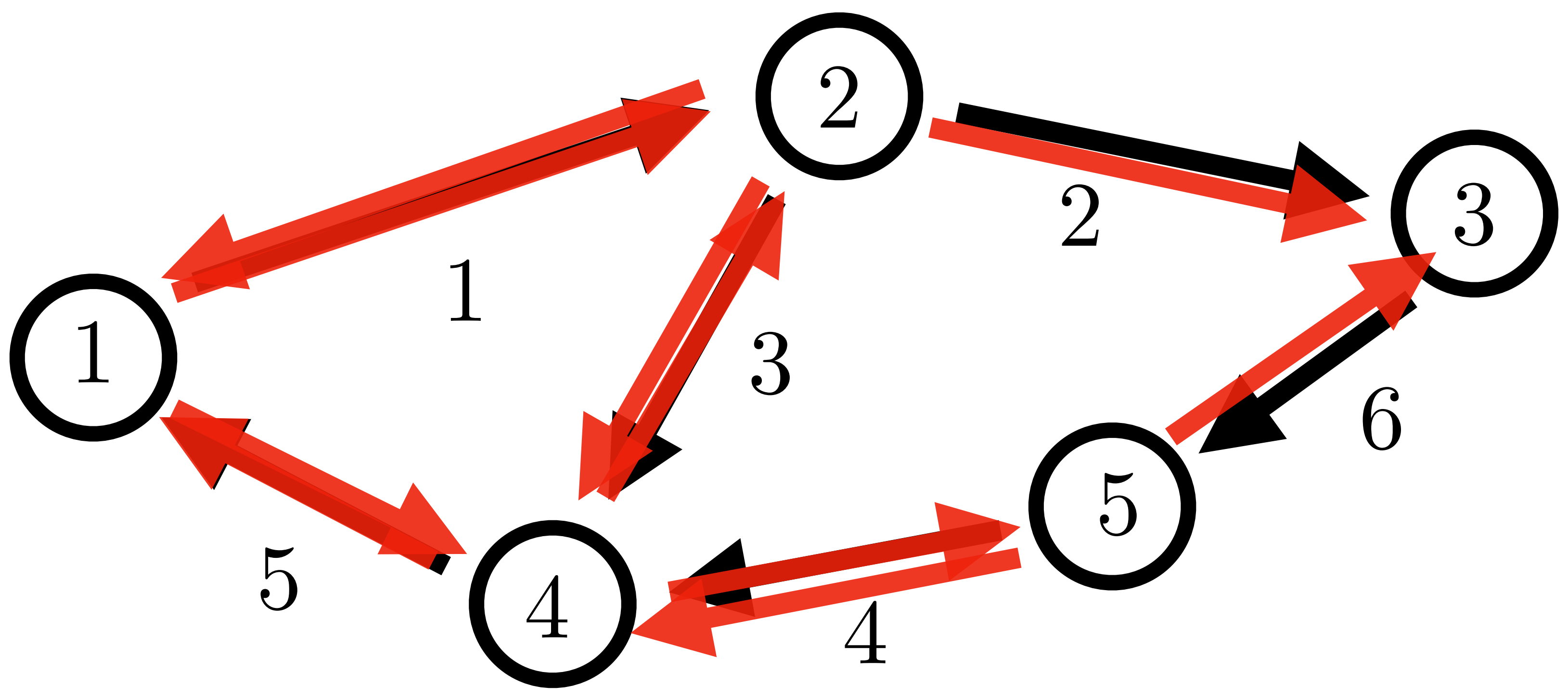
Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

Start @
node 4

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v|$ **diagonal**

Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ; \text{ if } e = (v, v') \in \mathcal{E} \\ 0 & ; \text{ otherwise} \end{cases}$

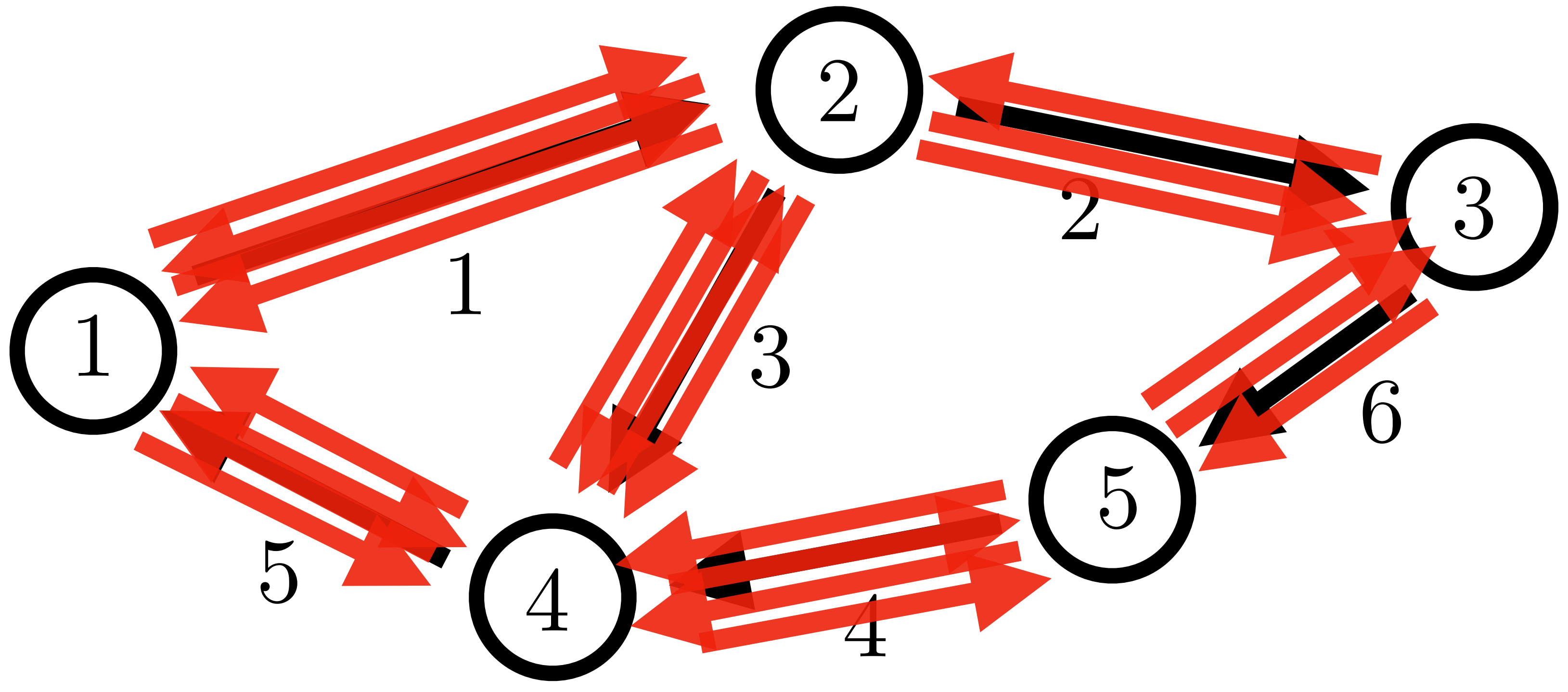
Powers of Adjacency

Thru Node 1 — Thru Node 2 — Thru Node 5

Start @ node 4

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

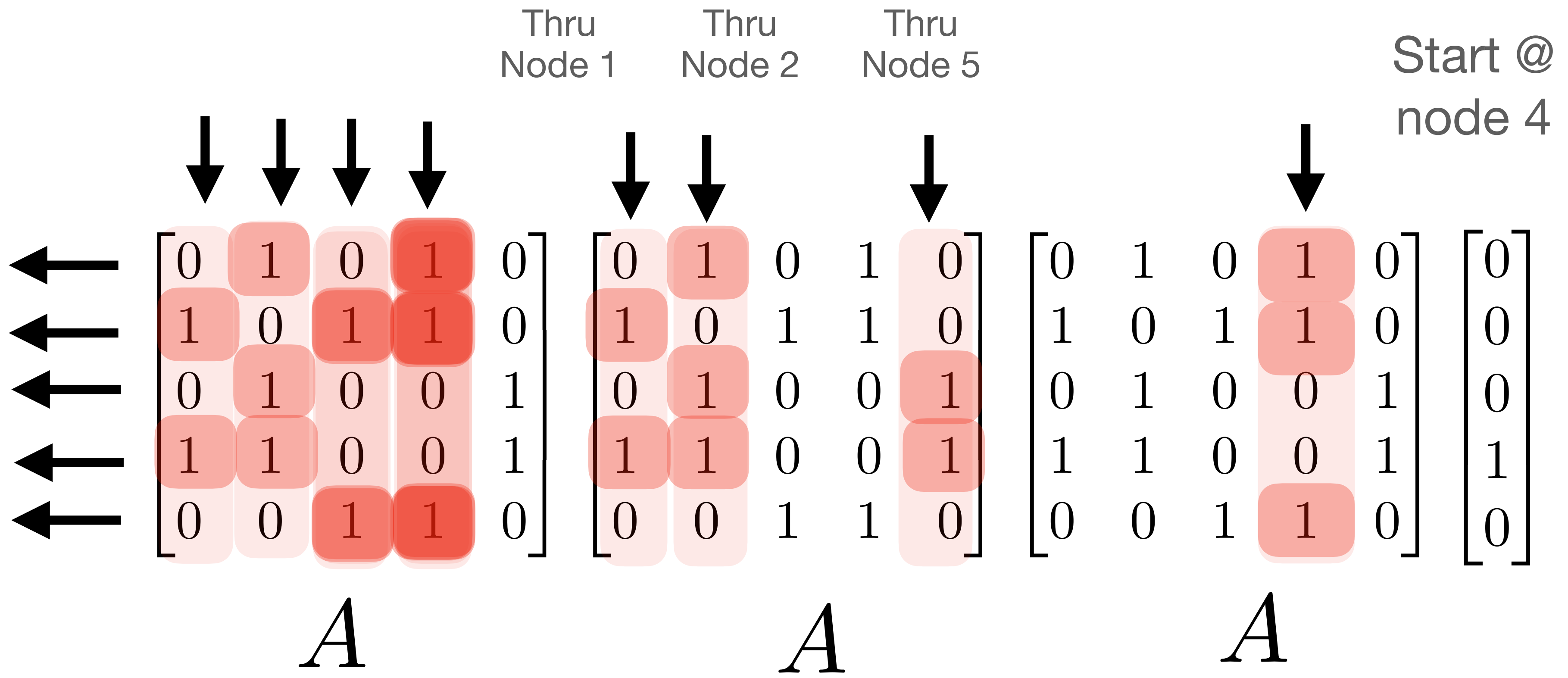
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\ \Delta\ ]_{vv} = |\mathcal{N}_v|$ $\quad$ **diagonal**

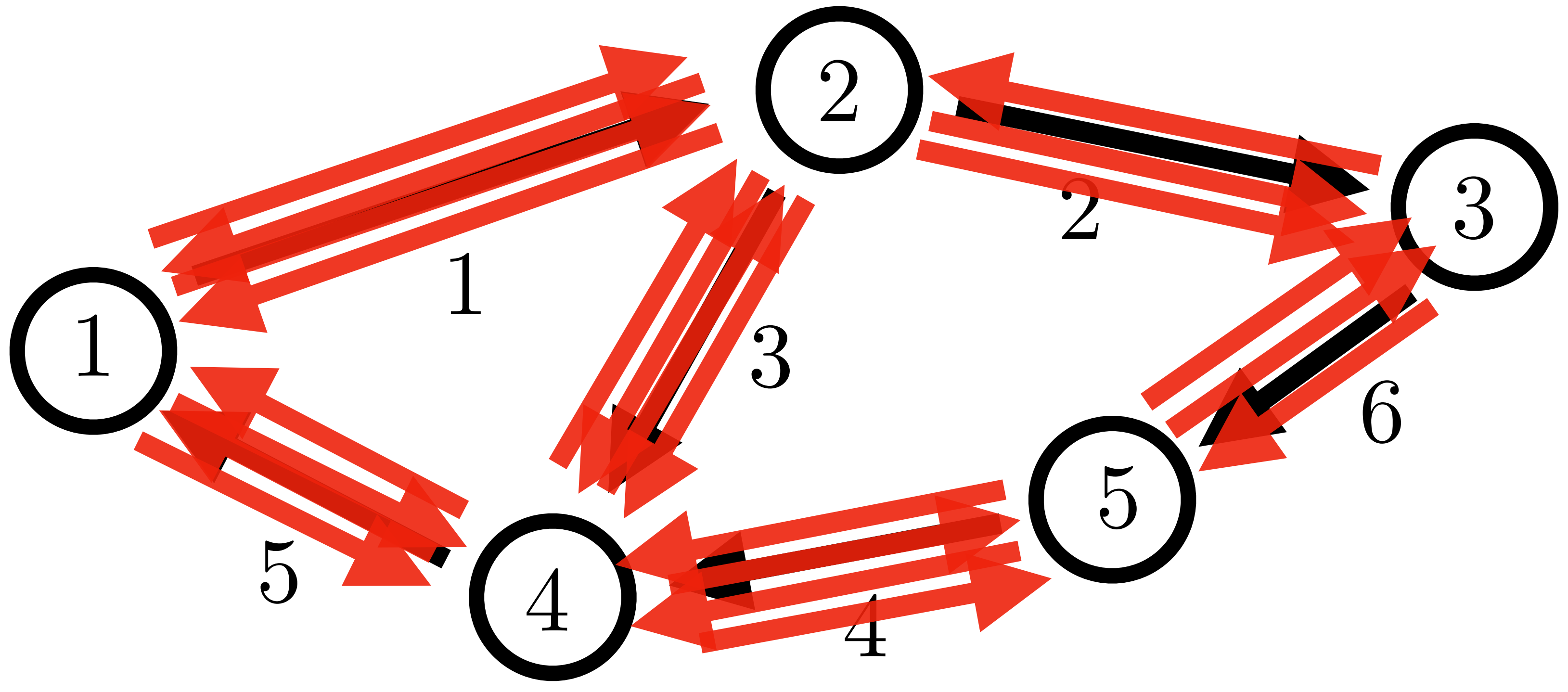
Adjacency Matrix $\quad [\ A\ ]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency



$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

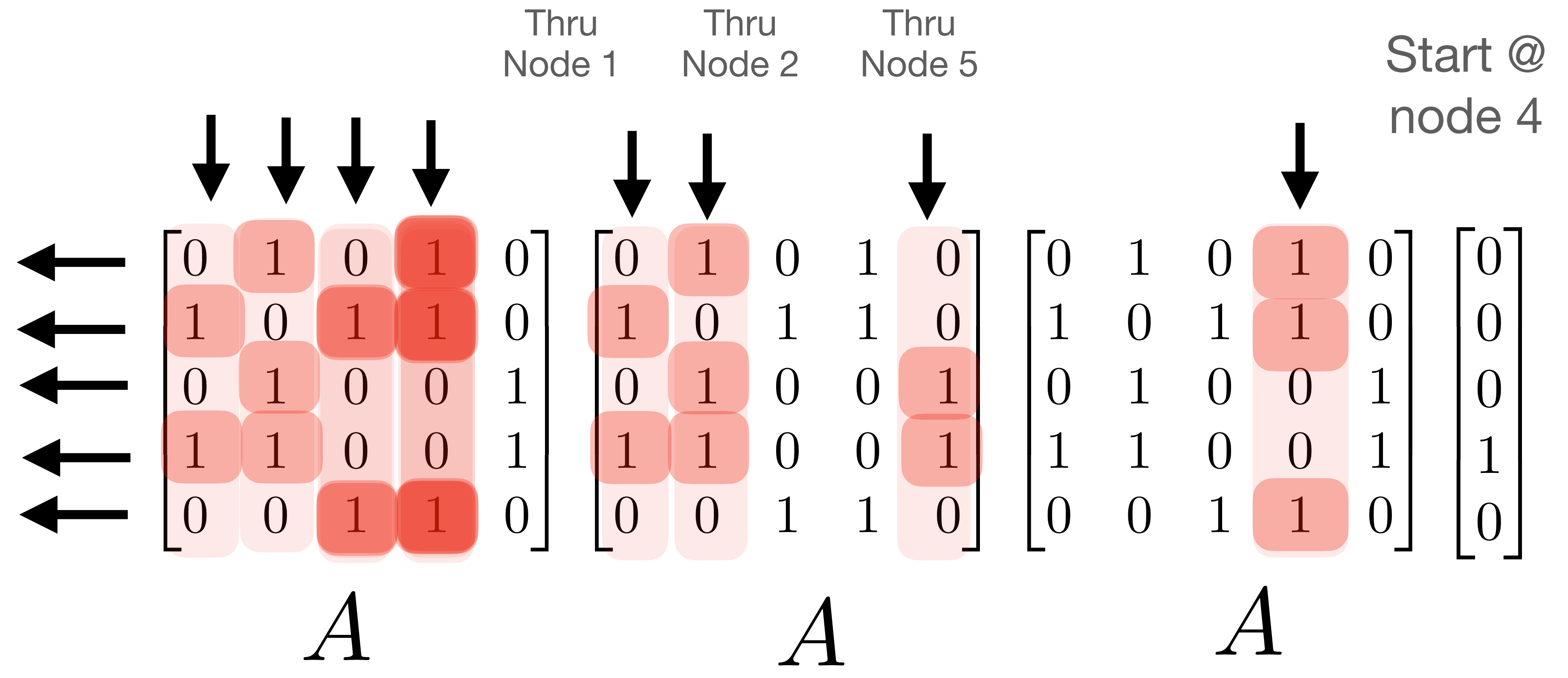
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v|$ $\quad$ **diagonal**

Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency



\# 3-step paths from node 4 to node 1

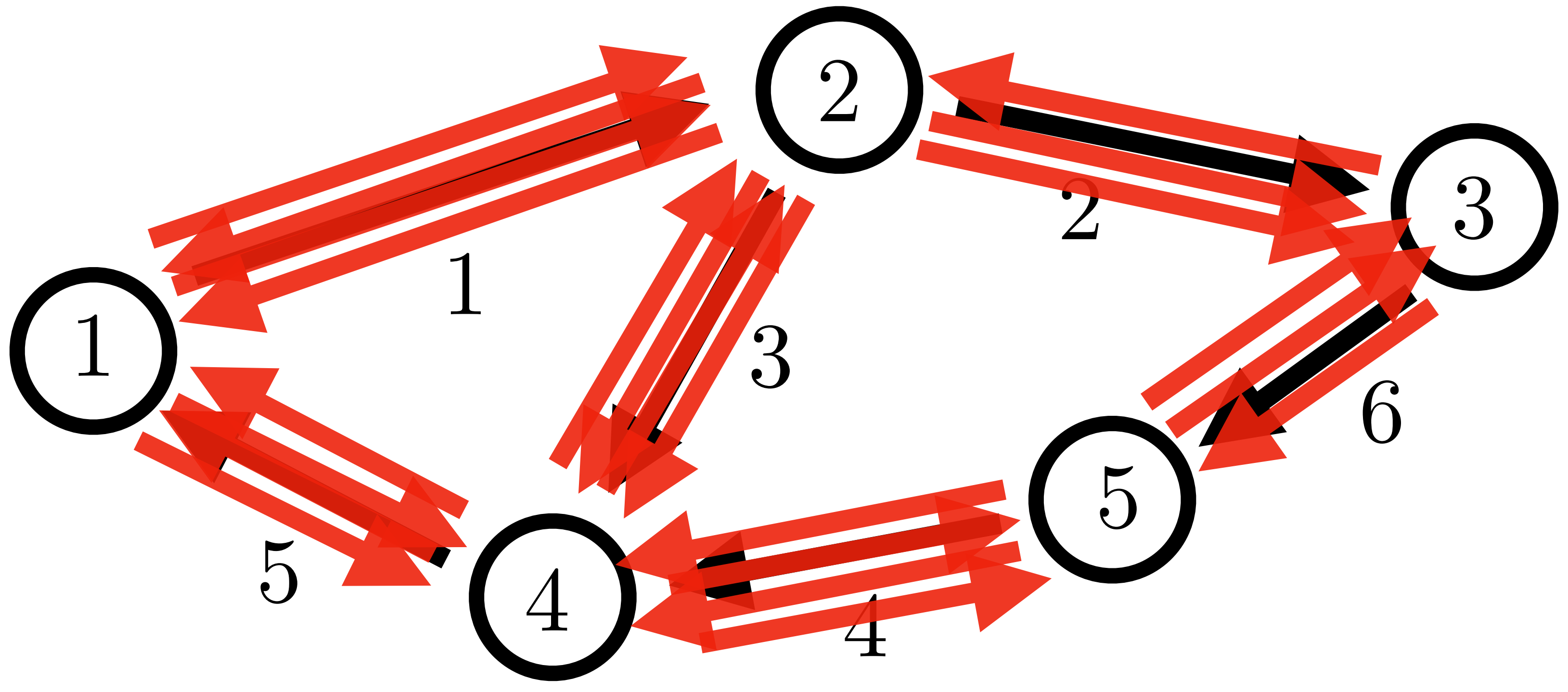
\# 3-step paths from node 4 to node 2

⋮

\# 3-step paths from node 4 to node 5

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Adjacency Matrix



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Laplacian $\quad L = DD^T = \Delta - A$

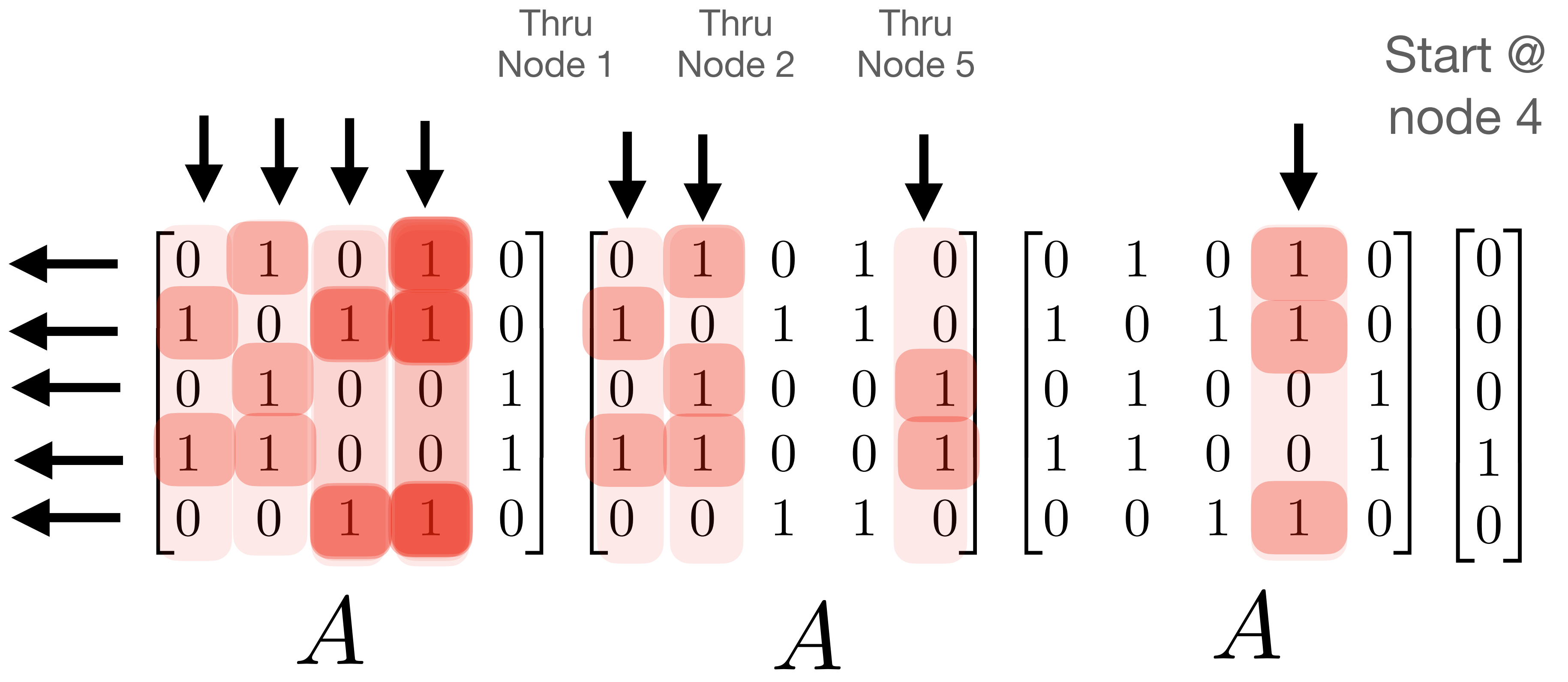
Degree Matrix $\quad [\,\Delta\,]_{vv} = |\mathcal{N}_v| \quad$ **diagonal**

Adjacency Matrix $\quad [\,A\,]_{vv'} = \begin{cases} 1 & ;\ \text{if } e = (v, v') \in \mathcal{E} \\ 0 & ;\ \text{otherwise} \end{cases}$

Powers of Adjacency

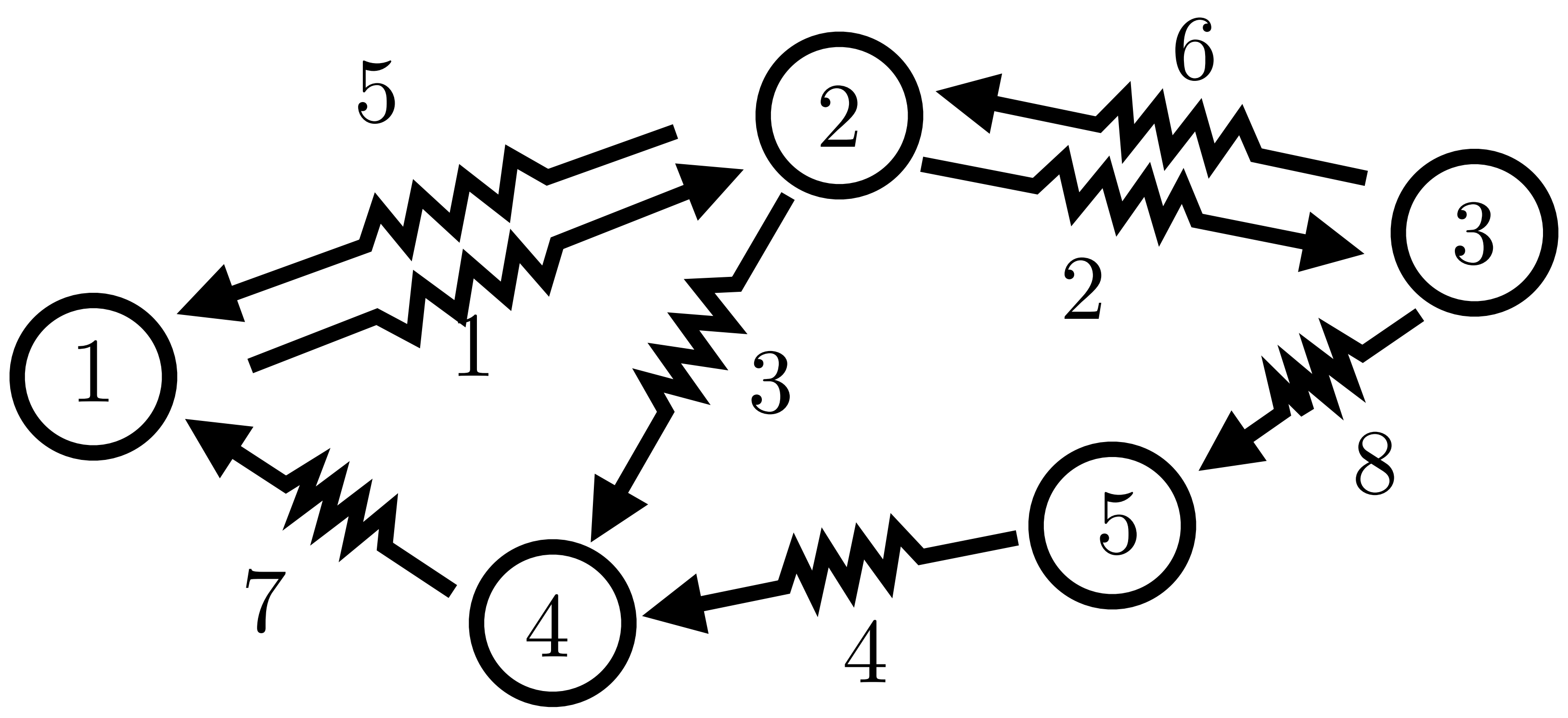
$$[\,A^k\,]_{vv'}$$

 # k-step paths from node v to node v’



$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $\quad D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

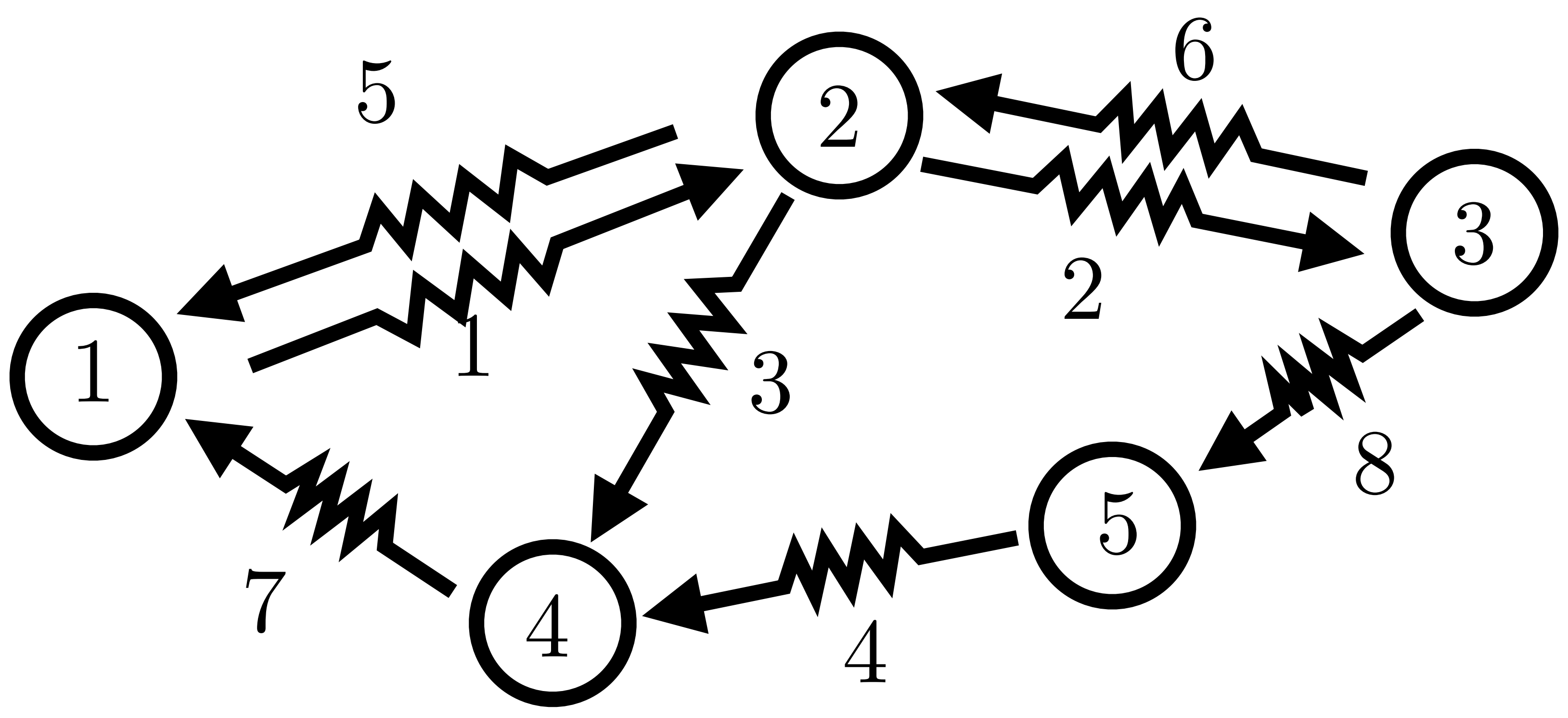
$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Weighted Laplacian $\quad L_W = DWD^T$

$$L_W = DWD^T = U_W \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} U_W^T$$

$$= \begin{bmatrix} | & | \\ U'_W & \bar{\mathbf{1}} \\ | & | \end{bmatrix} \begin{bmatrix} \Sigma_W^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} - & {U'}_W^T & - \\ - & \bar{\mathbf{1}}^T & - \end{bmatrix}^T \qquad \bar{\mathbf{1}} = \begin{bmatrix} \mathbf{1}_{|\mathcal{V}_1|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \mathbf{1}_{|\mathcal{V}_k|} \end{bmatrix}$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Weighted Laplacian $L_W = DWD^T$

Action:

$$L_W u = \underbrace{\left[\; D \;\right] \underbrace{\left[\; W \;\right]\left[\; D^T \;\right]\begin{bmatrix} | \\ u \\ | \end{bmatrix}}_{\text{...tension created in edges scaled by weights}}}_{\text{… summed resulting tension on nodes}}$$

"heights"
of nodes

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

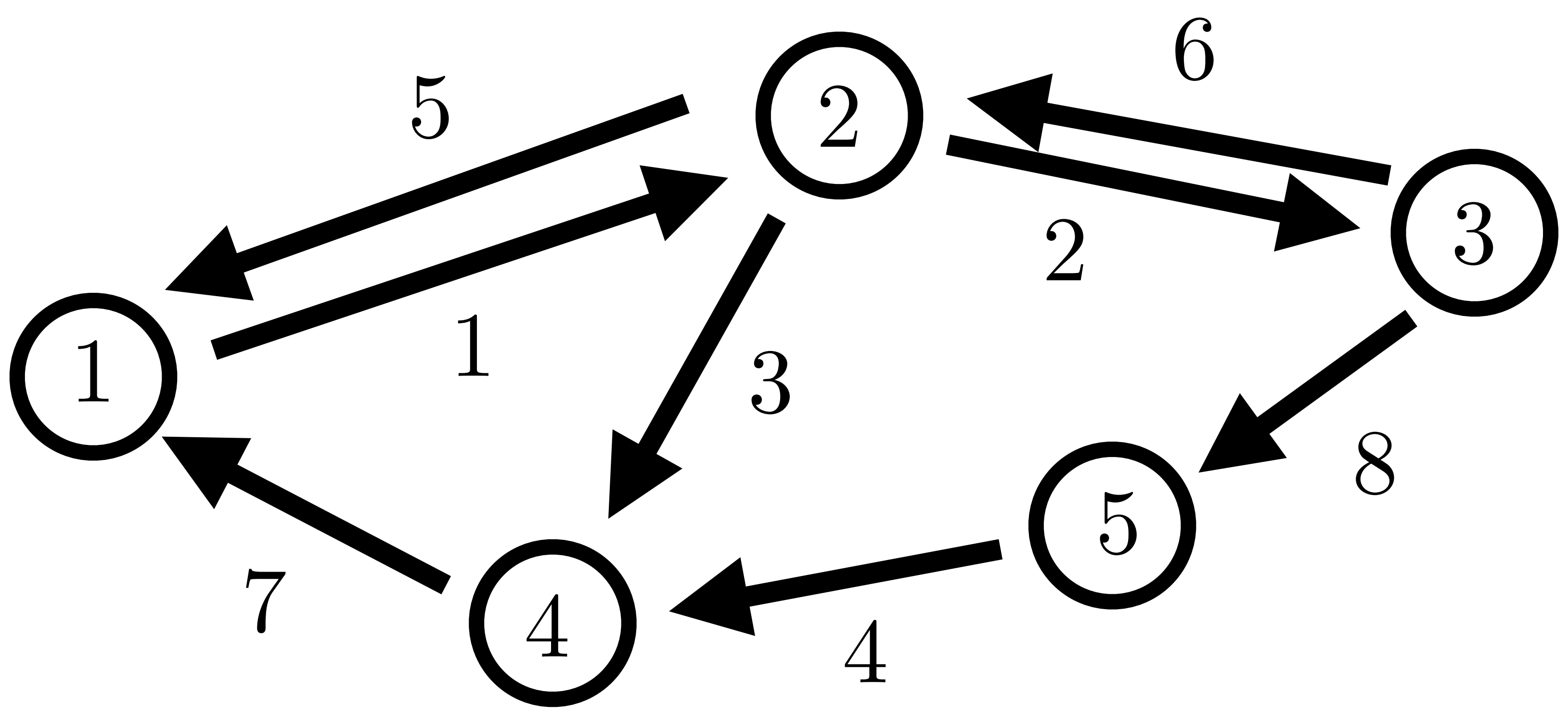
Laplacian row "shape" matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col "shape" matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Edge Laplacian $L_e = D^T D$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

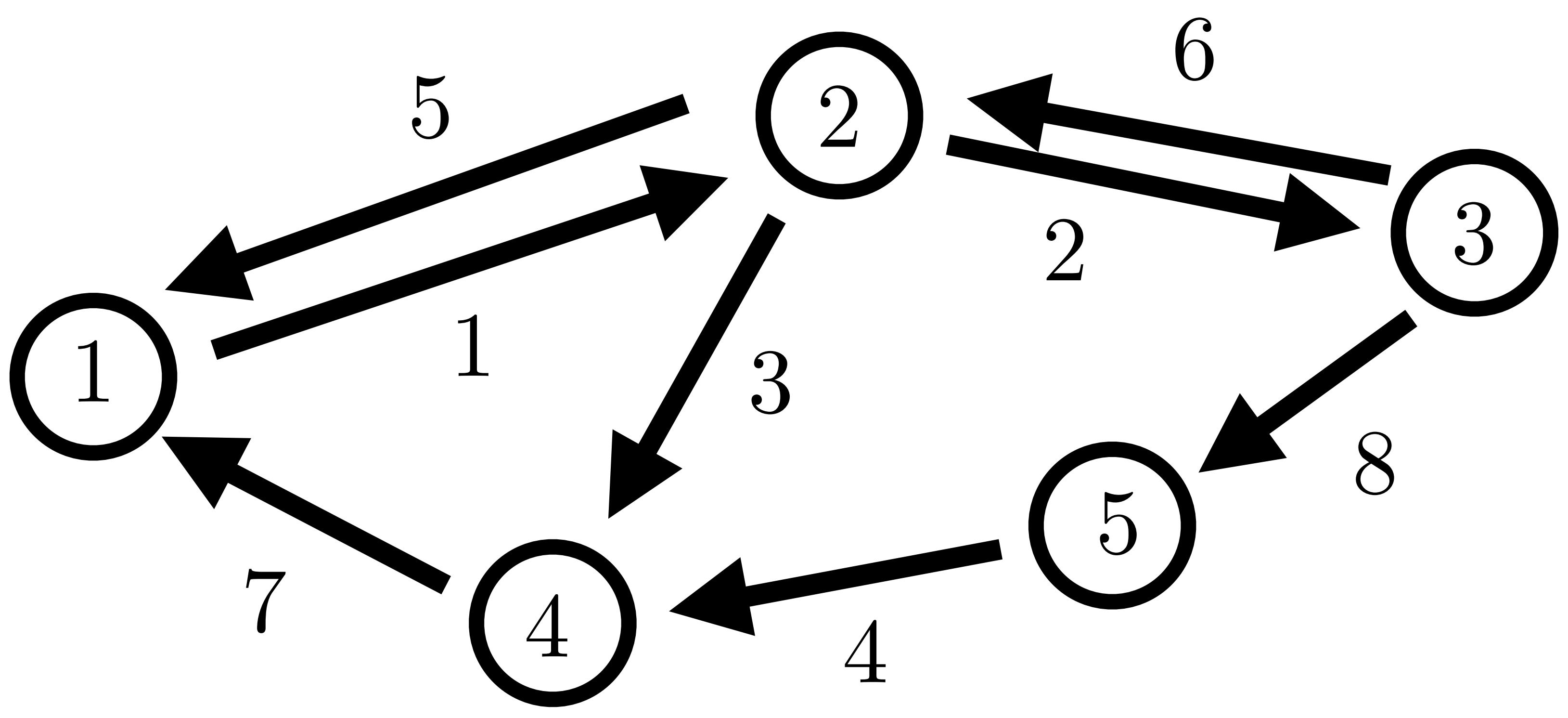
Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Graph Laplacians



Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

Incidence Matrix: $D \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$

$$D = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Incidence SVD $D = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$

Laplacian row “shape” matrix (squared)

$$L = DD^T = U \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} U^T$$

Edge-Laplacian col “shape” matrix (squared)

$$L_e = D^T D = V \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Edge Laplacian $L_e = D^T D$

Action:

$$L_e \tau = \underbrace{\left[\ D^T\ \right] \underbrace{\left[\ D\ \right] \begin{bmatrix} | \\ \tau \\ | \end{bmatrix}}_{\text{…summed tension on nodes}}}_{\text{… differential in tension along edges}}$$

“Tension” in edges

REVIEW: Nullspace - Column Geometry (Computation)

$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns

Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \end{bmatrix}$$

$$A'' = A'B$$

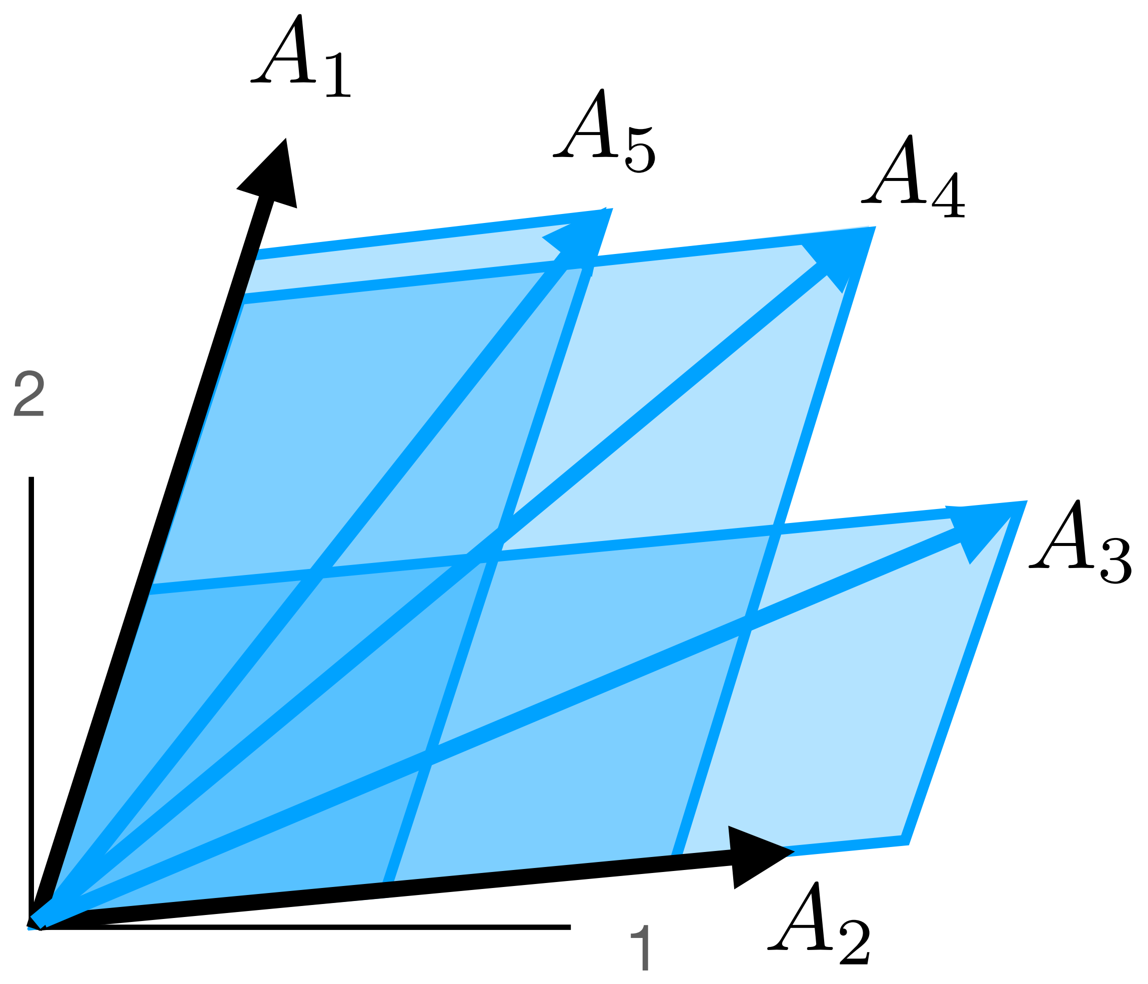
$$A = \begin{bmatrix} A' & A'B \end{bmatrix} = A' \begin{bmatrix} I & B \end{bmatrix}$$

$$\begin{bmatrix} | & | & | \\ A_3 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1B_{13} + A_2B_{23} & A_1B_{14} + A_2B_{24} & A_1B_{15} + A_2B_{25} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{23} & B_{24} & B_{25} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$



PROOF:

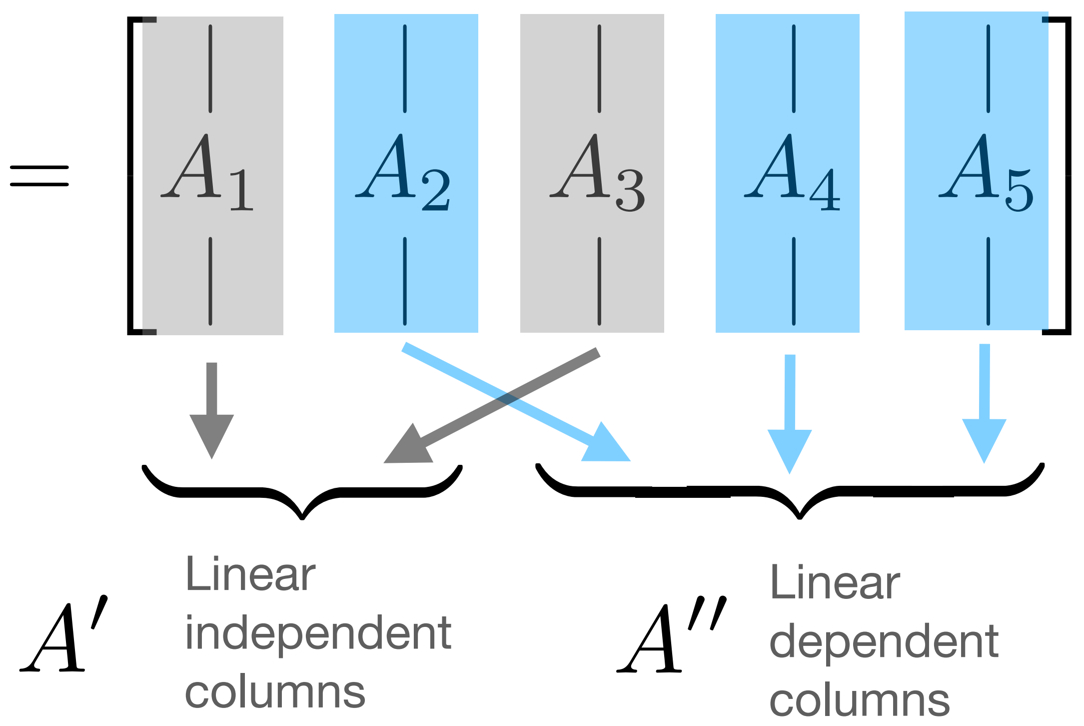
Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \overset{A' \text{ lin. ind.}}{\Rightarrow} \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

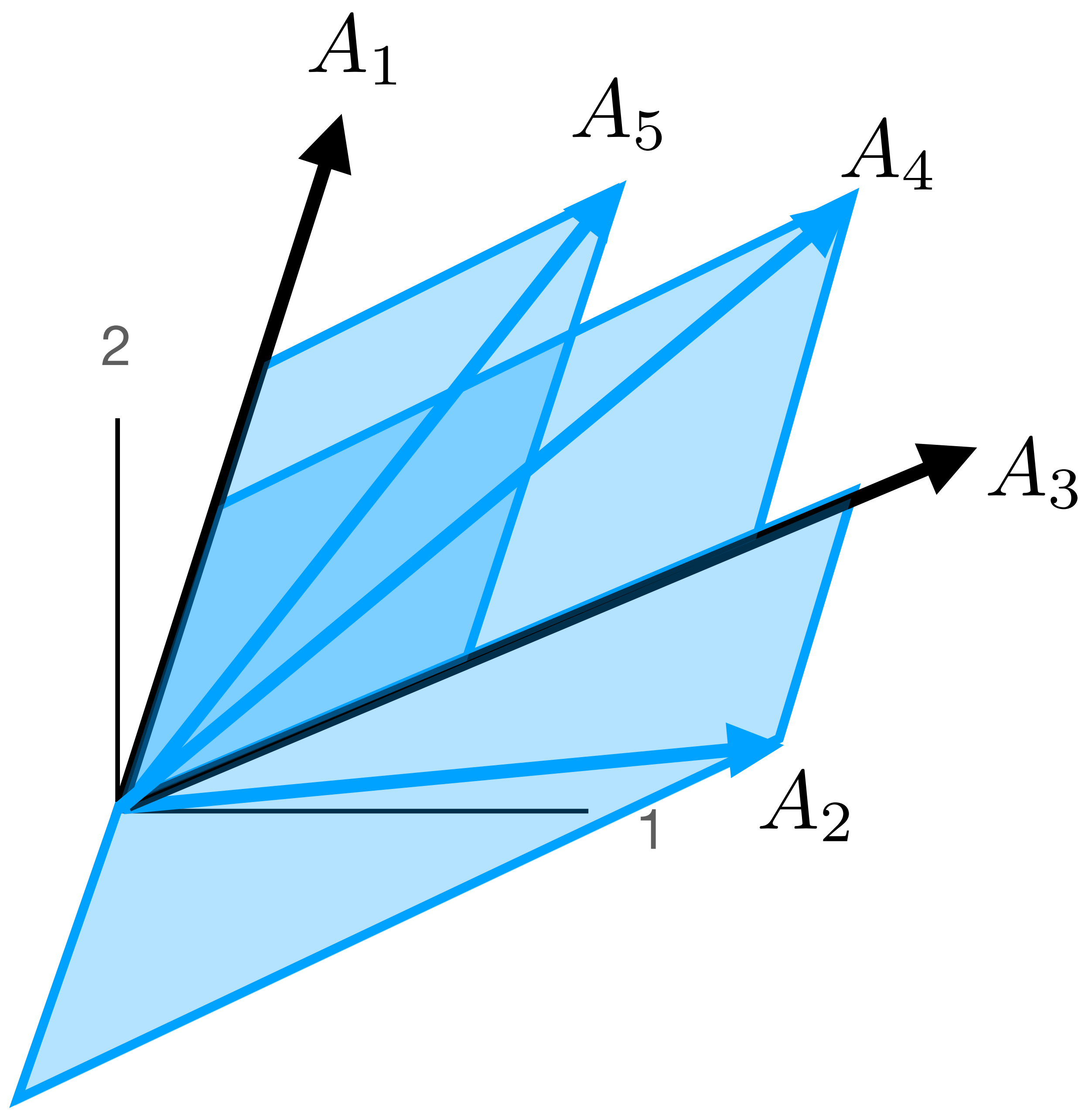
REVIEW: Nullspace - Column Geometry (Computation)



$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ B_{33} & B_{34} & B_{35} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_4 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_3 B_{32} & A_1 B_{14} + A_3 B_{34} & A_1 B_{15} + A_3 B_{35} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ B_{33} & B_{34} & B_{35} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

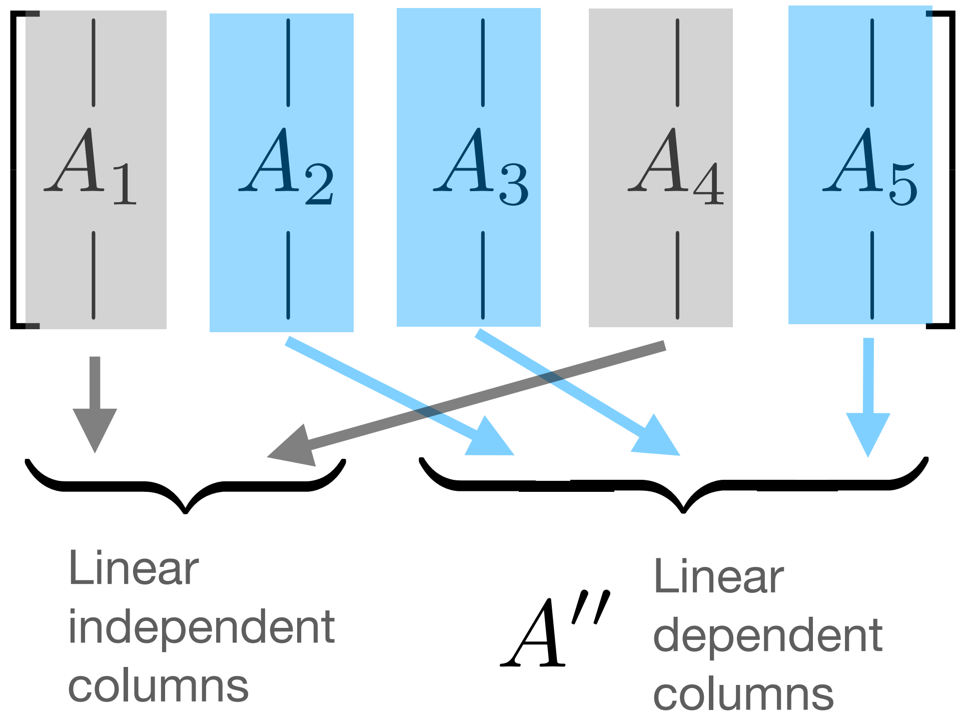
Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \Rightarrow z = 0$

Span: $x \in \mathcal{N}(A)$

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \overset{A' \text{ lin. ind.}}{\Rightarrow} x' + Bx'' = 0$$

$$\Rightarrow x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$

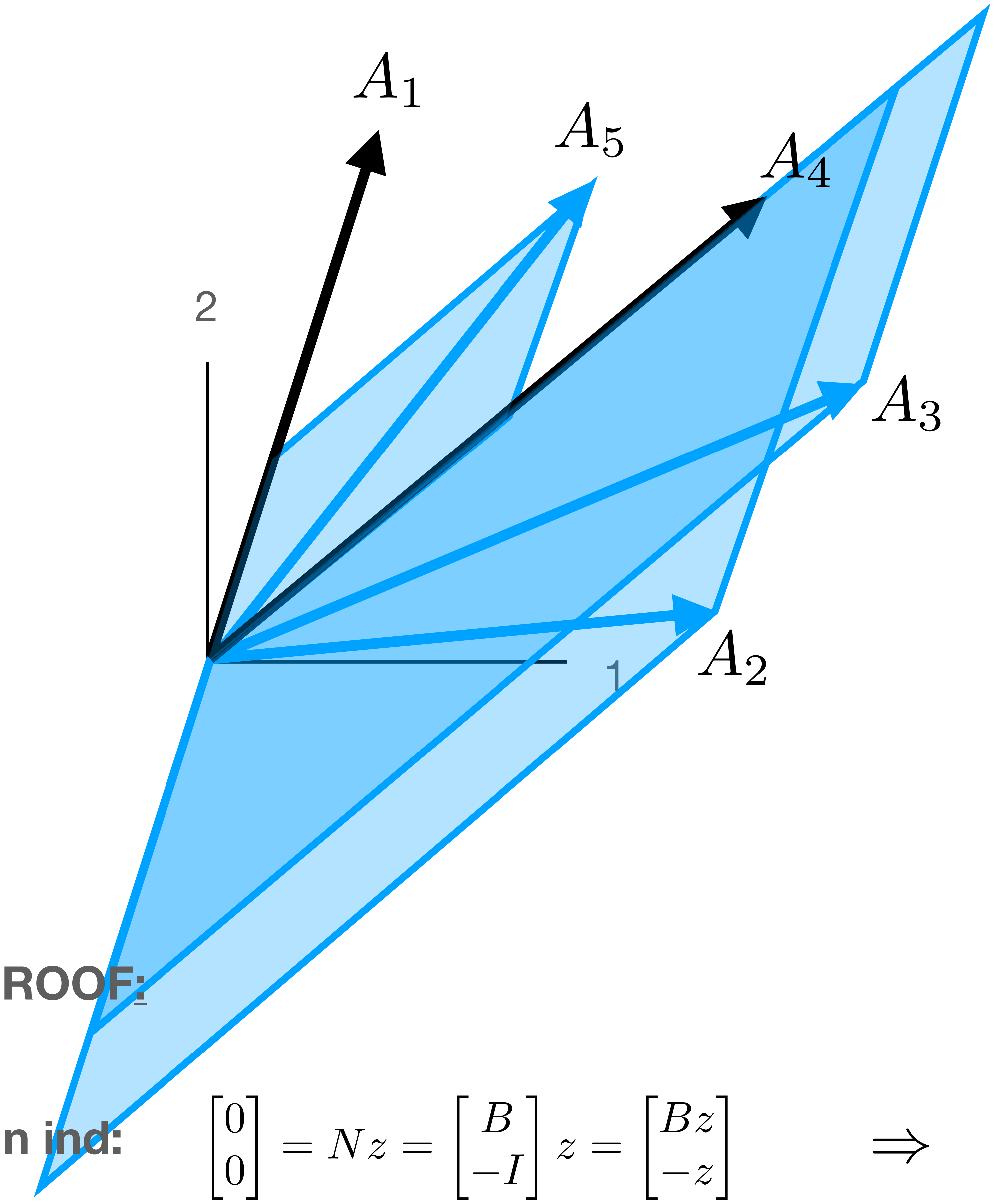
REVIEW: Nullspace - Column Geometry (Computation)



$$A = \begin{bmatrix} A' & A'' \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ A_1 & A_2 & A_3 & A_4 & A_5 \\ | & | & | & | & | \end{bmatrix}$$

A' Linear independent columns

A'' Linear dependent columns



Coordinates of linear dependent columns:

$$B = \begin{bmatrix} B_{12} & B_{13} & B_{15} \\ B_{42} & B_{43} & B_{45} \end{bmatrix}$$

$$A'' = A'B$$

$$A = \begin{bmatrix} A' & A'B \end{bmatrix} P = A' \begin{bmatrix} I & B \end{bmatrix} P$$

$$\begin{bmatrix} | & | & | \\ A_2 & A_3 & A_5 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A_1 B_{12} + A_4 B_{42} & A_1 B_{13} + A_4 B_{43} & A_1 B_{15} + A_4 B_{45} \\ | & | & | \end{bmatrix}$$

Nullspace basis:

$$N = P^{-1} \begin{bmatrix} B \\ -I \end{bmatrix} = \begin{bmatrix} B_{13} & B_{14} & B_{15} \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ B_{43} & B_{44} & B_{45} \\ 0 & 0 & -1 \end{bmatrix}$$

$$AN = 0$$

PROOF:

Lin ind: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = Nz = \begin{bmatrix} B \\ -I \end{bmatrix} z = \begin{bmatrix} Bz \\ -z \end{bmatrix} \quad \Rightarrow \quad z = 0$

Span: $x \in \mathcal{N}(A)$

A' lin. ind.

$$Ax = \begin{bmatrix} I & B \end{bmatrix} \begin{bmatrix} x' \\ x'' \end{bmatrix} = A'(x' + Bx'') \quad \Rightarrow \quad x' + Bx'' = 0$$

$$\Rightarrow \quad x = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} B \\ -I \end{bmatrix} (-x'')$$