Transition Kernels, Markov Chains
Markov Decision Processes

Algebraic Graph Theory

Acknowledgements:
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Sarah Li
Yue Yu
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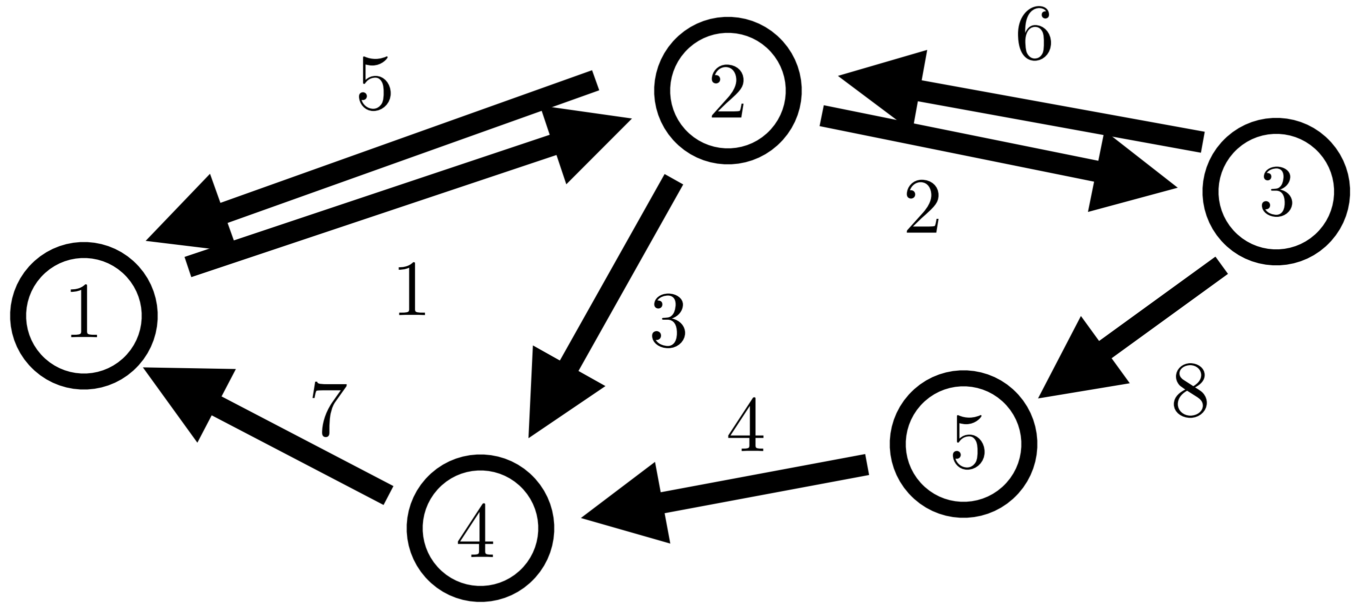
Spring 2022 - Dan Calderone

Probabilistic Transitions



Graph:

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Vertices $v \in \mathcal{V}$

Edges $e \in \mathcal{E}$ $\quad e = (v, v')$

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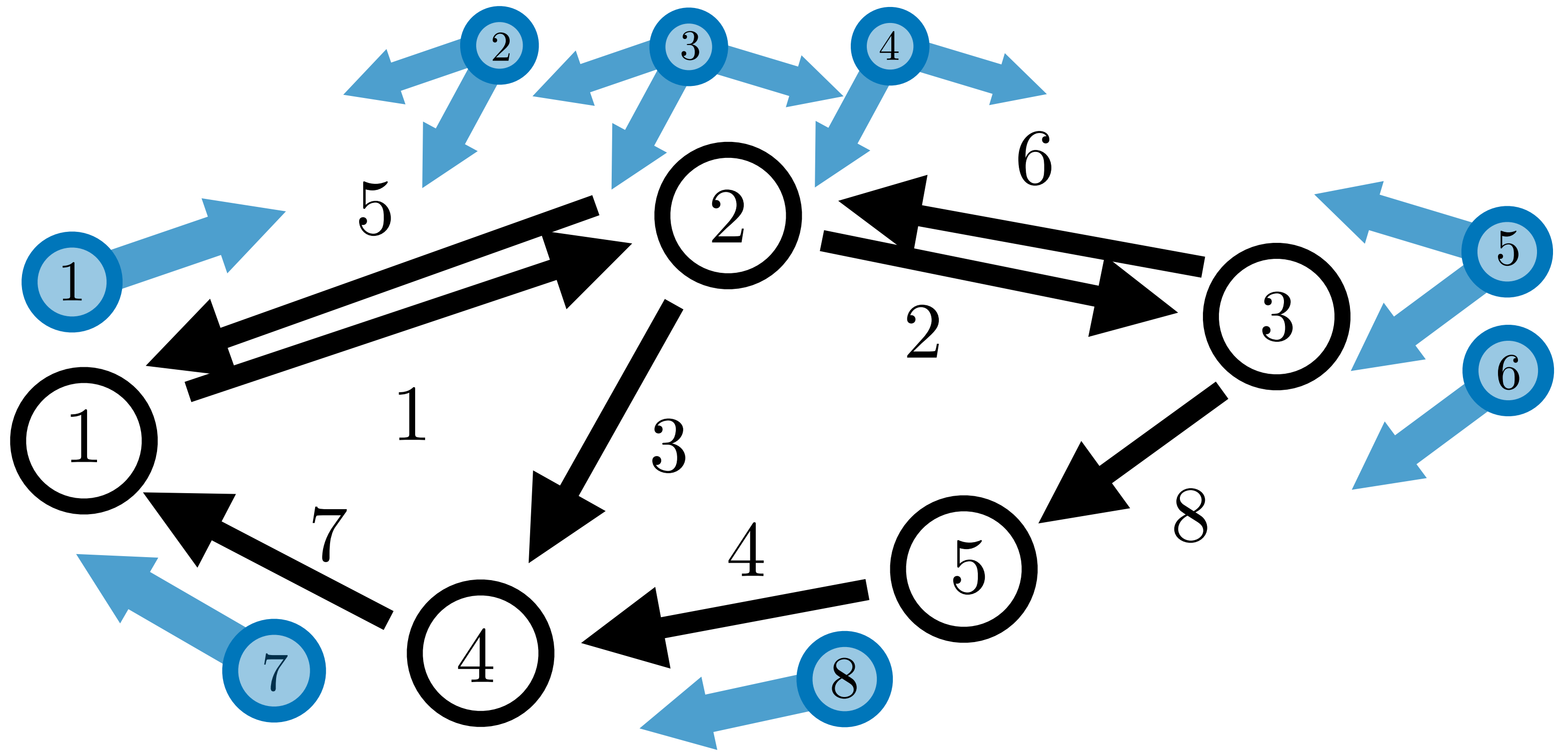

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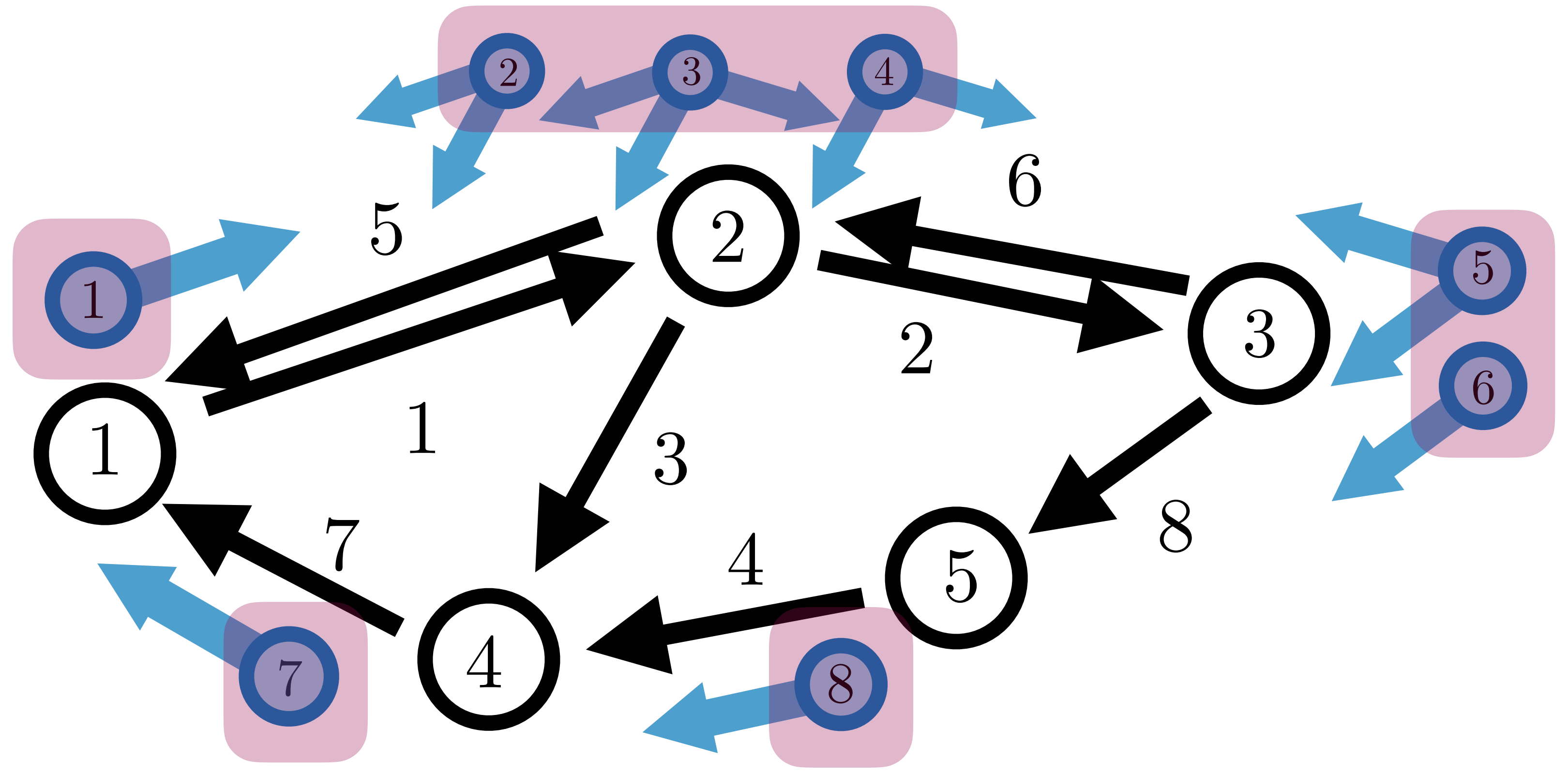

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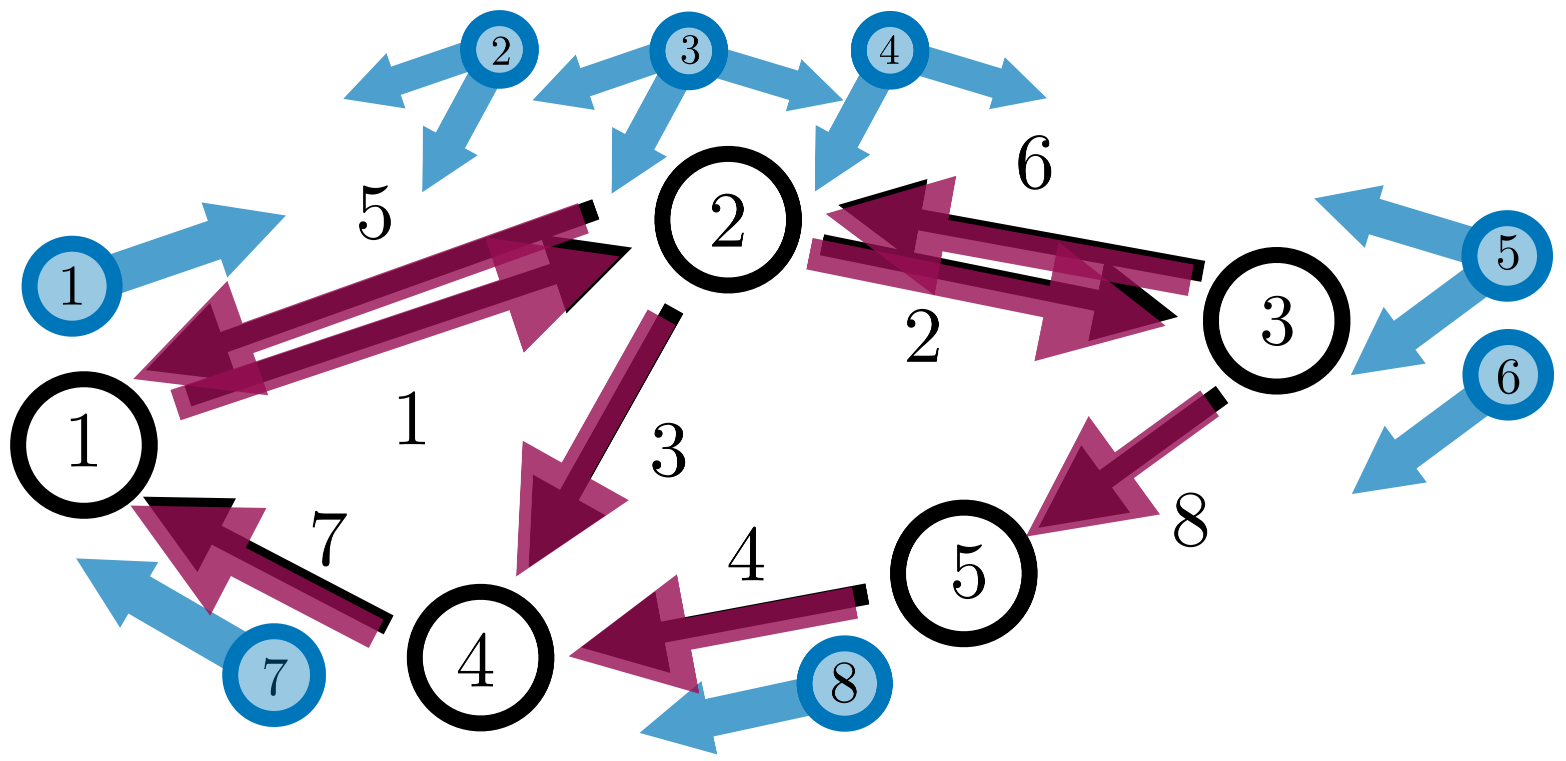

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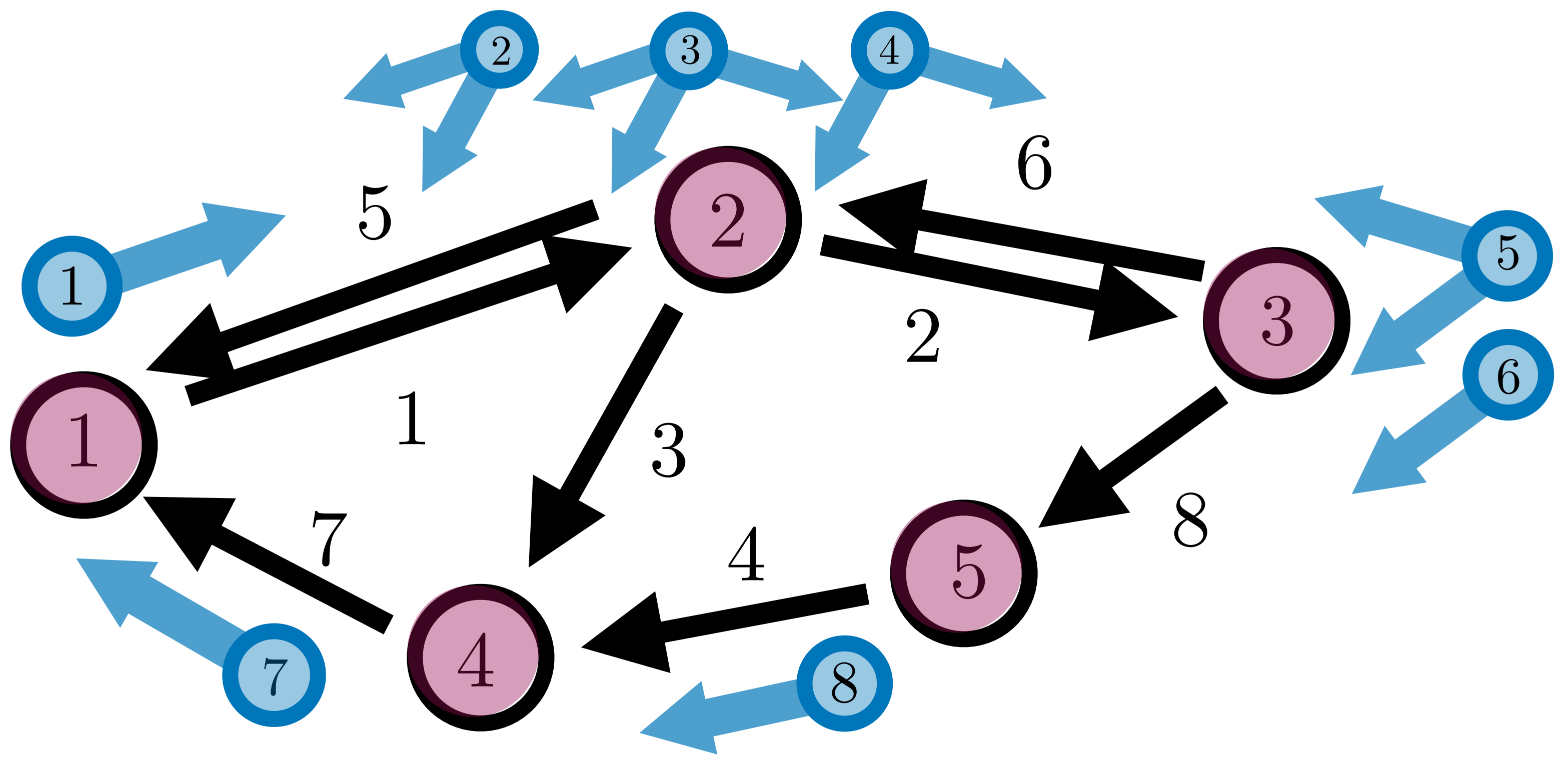

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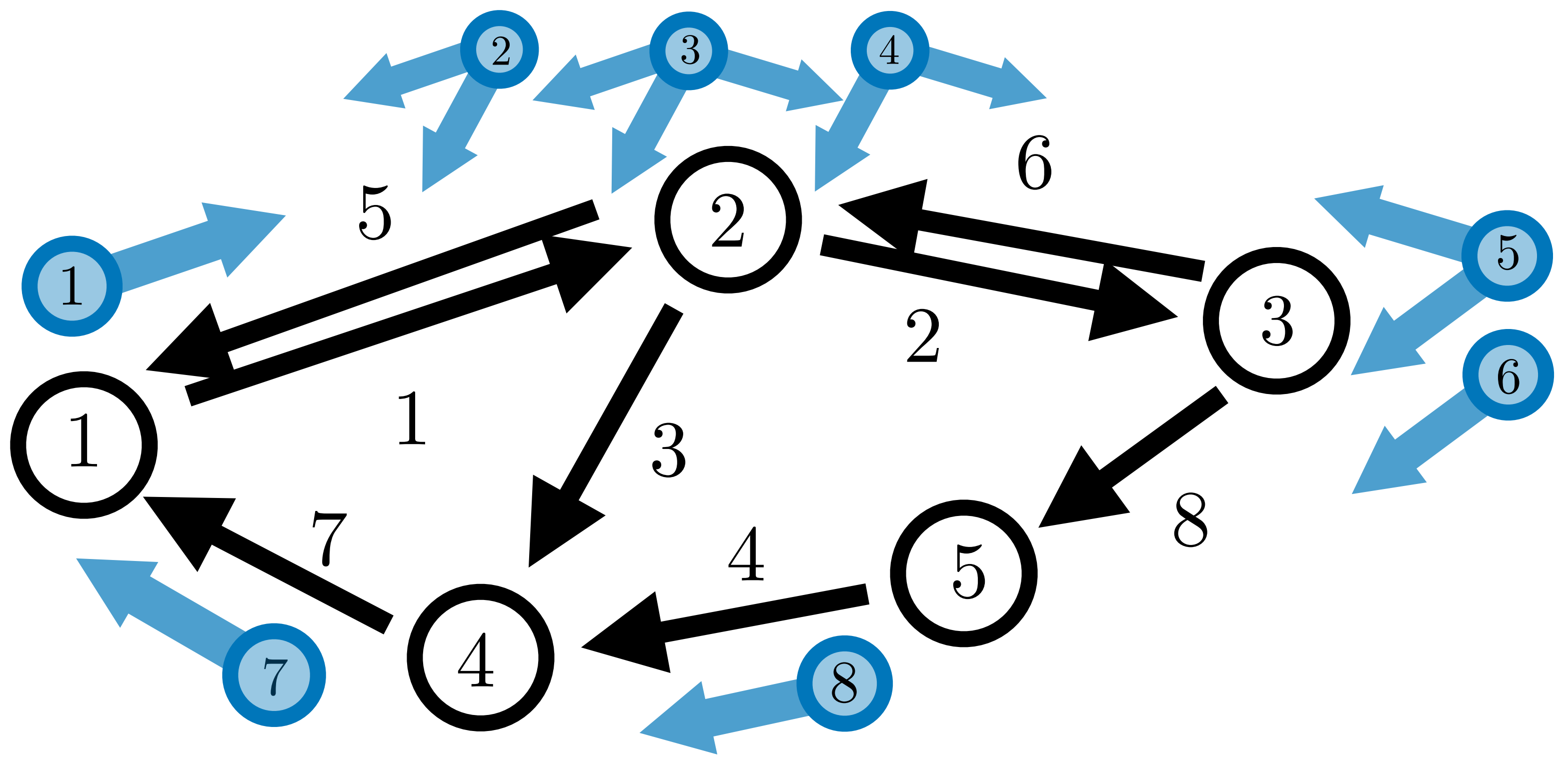

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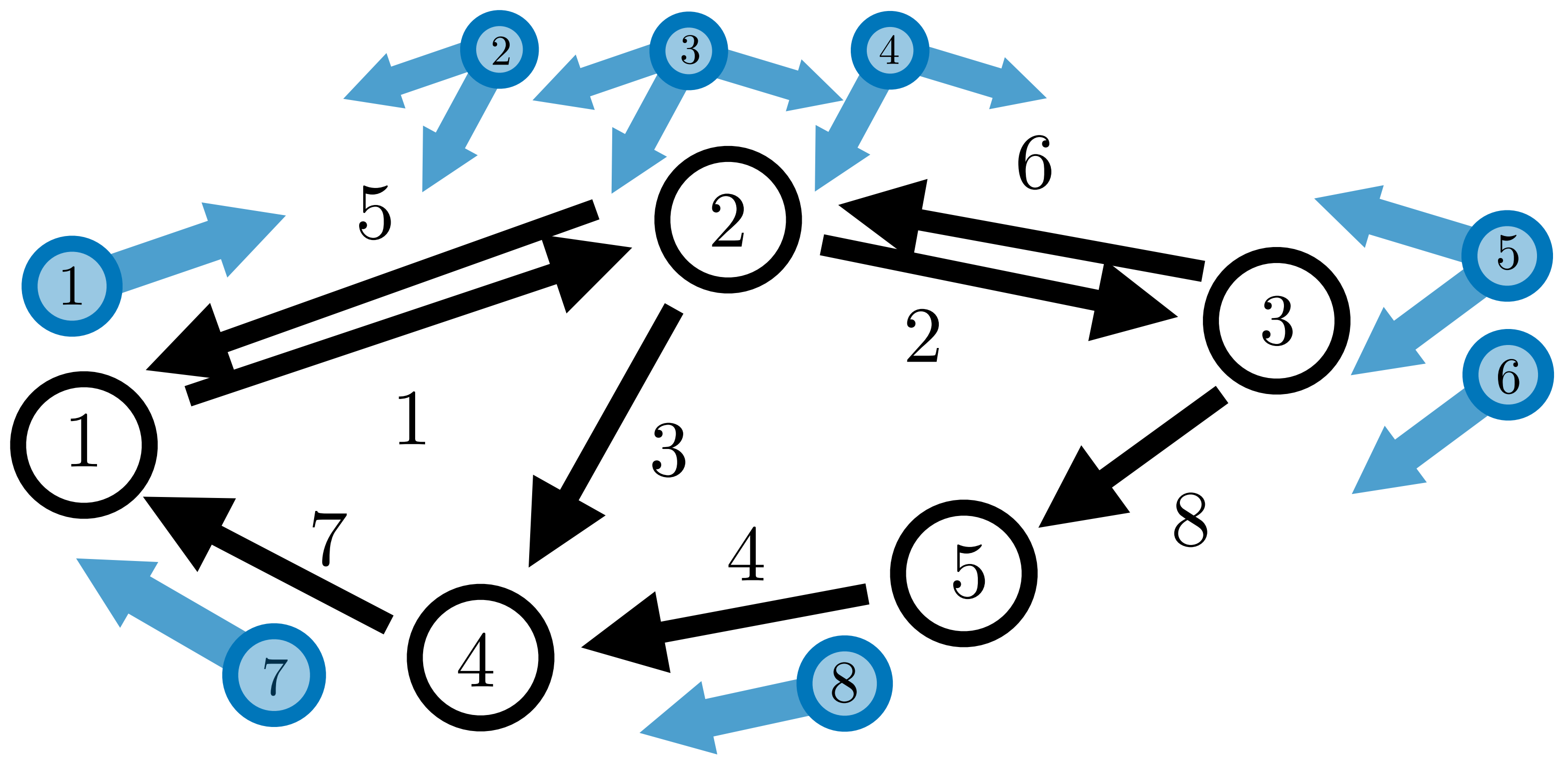

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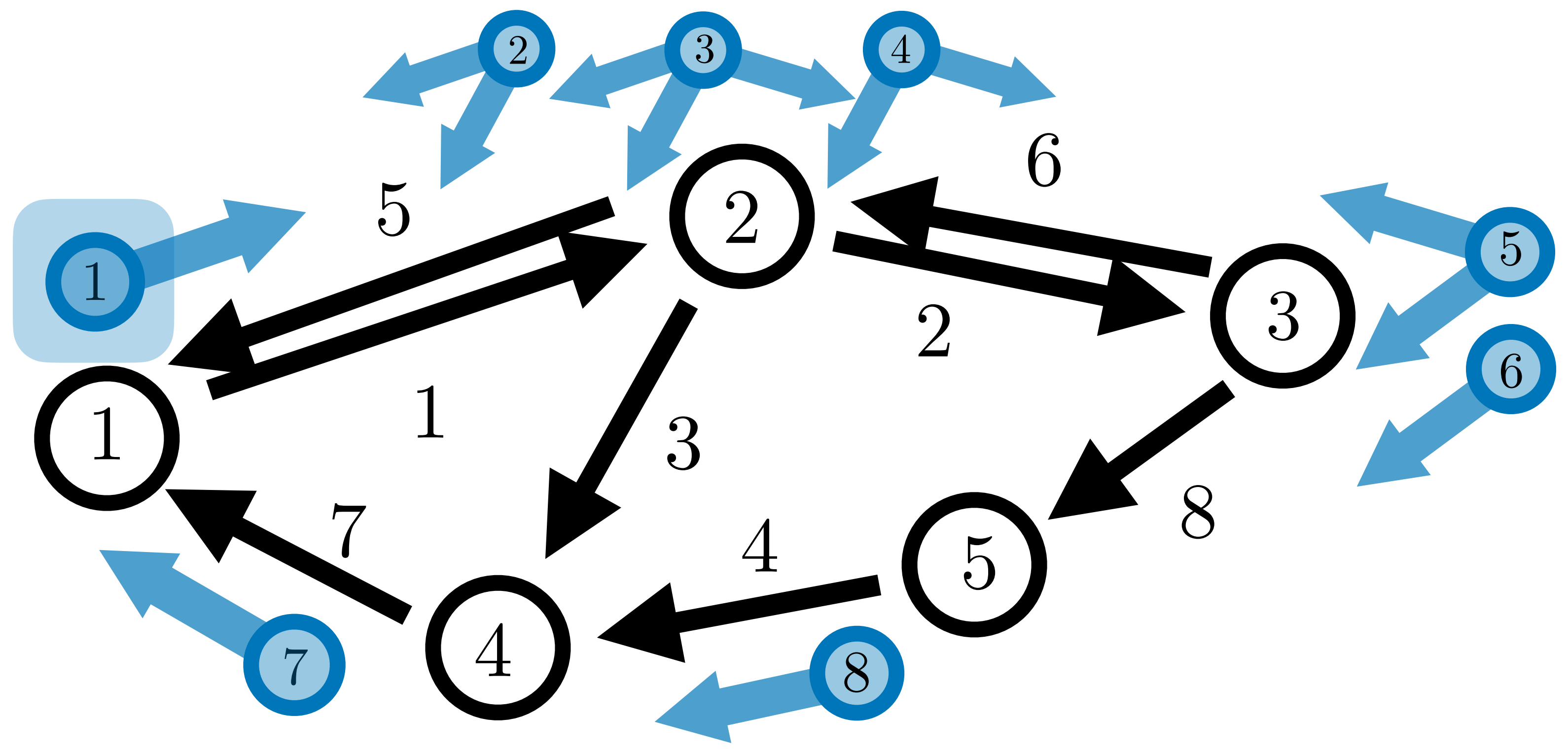

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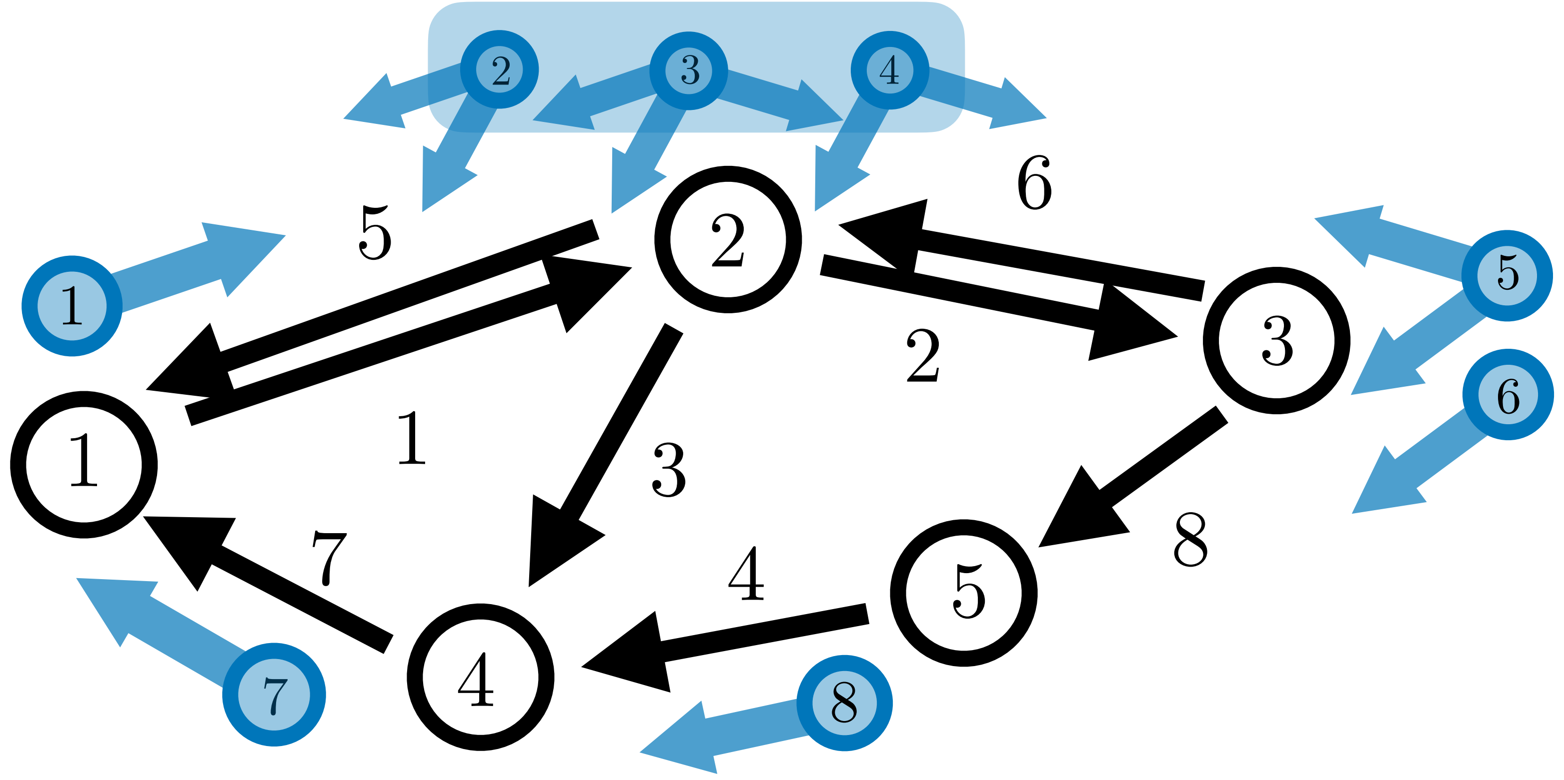

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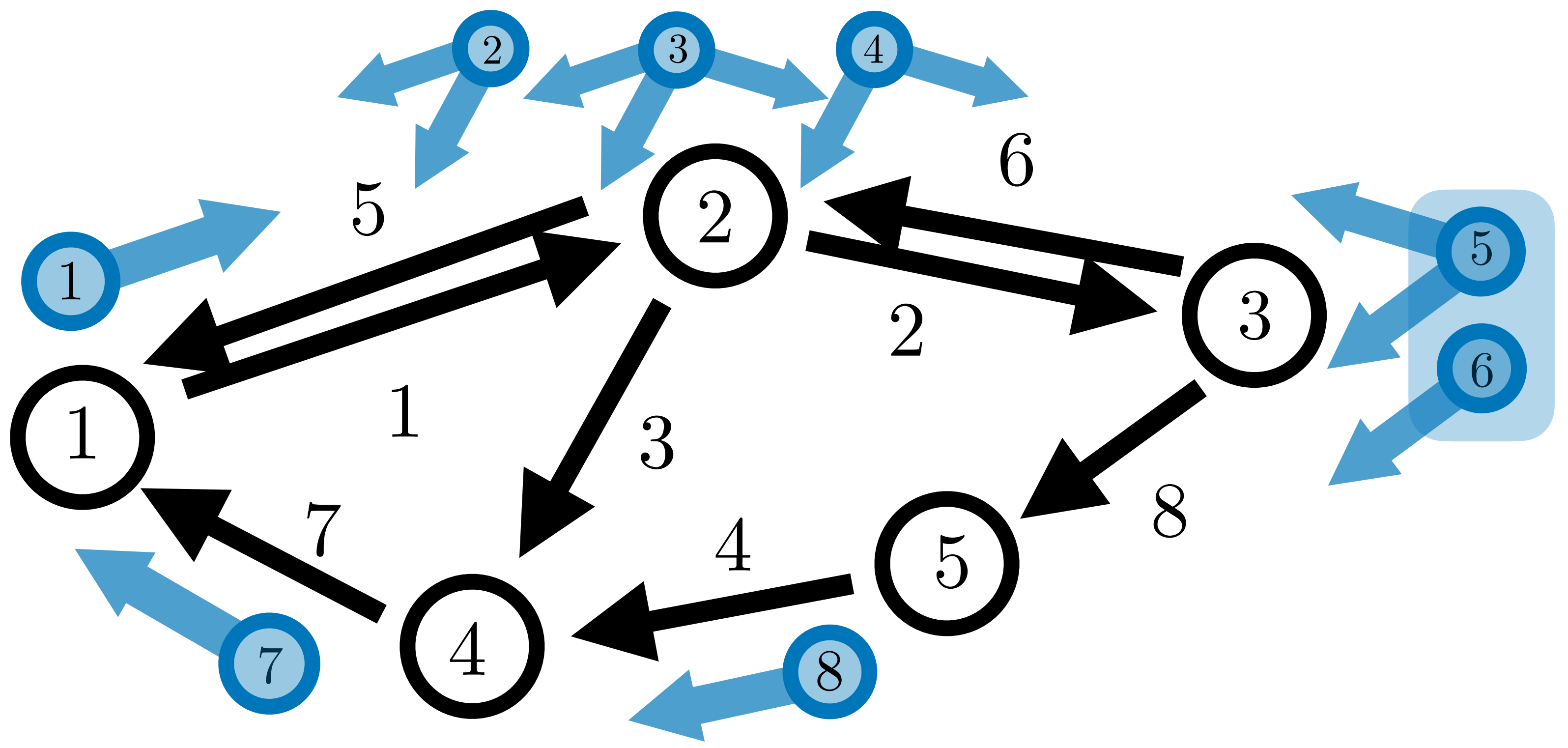

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$$W = \begin{array}{cc} & \text{actions} \\ \text{edges} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.5 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Transition Kernel

Graph:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

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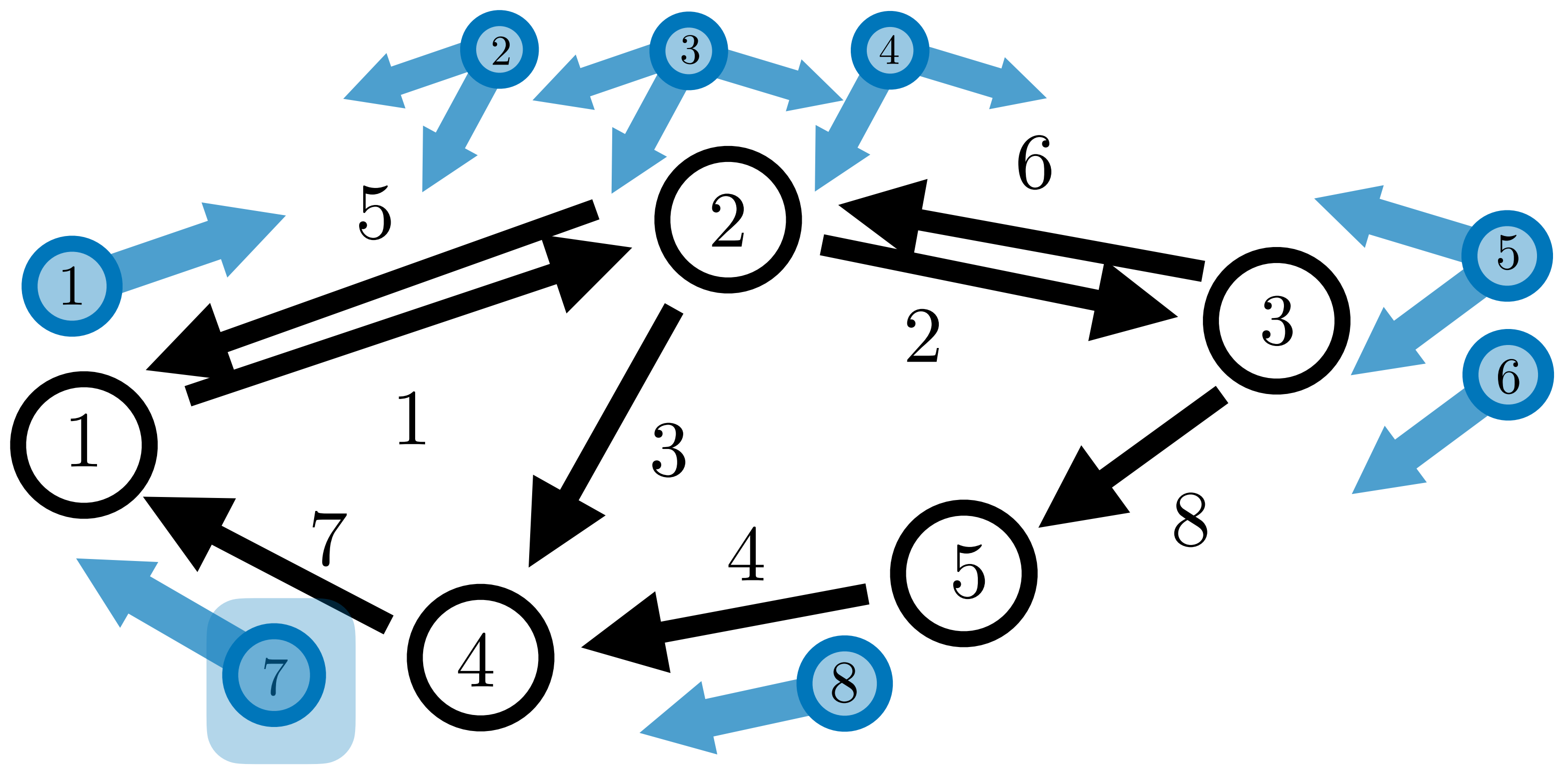

Incidence Matrices

$$E \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|} \quad E = E_{\text{in}} - E_{\text{out}}$$

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Mass conservation

$$\left[\ E_{\mathcal{A}} - P\ \right]x = \left[\ E_{\text{out}} - E_{\text{in}}\ \right]Wx = 0$$

Transition Kernel

$$P \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$$
$$W \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{A}|}$$

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...action to edge

$$E_{\mathcal{A}} = \begin{array}{cc} & \text{actions} \\ \text{states} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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Incidence Matrices

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...action to state

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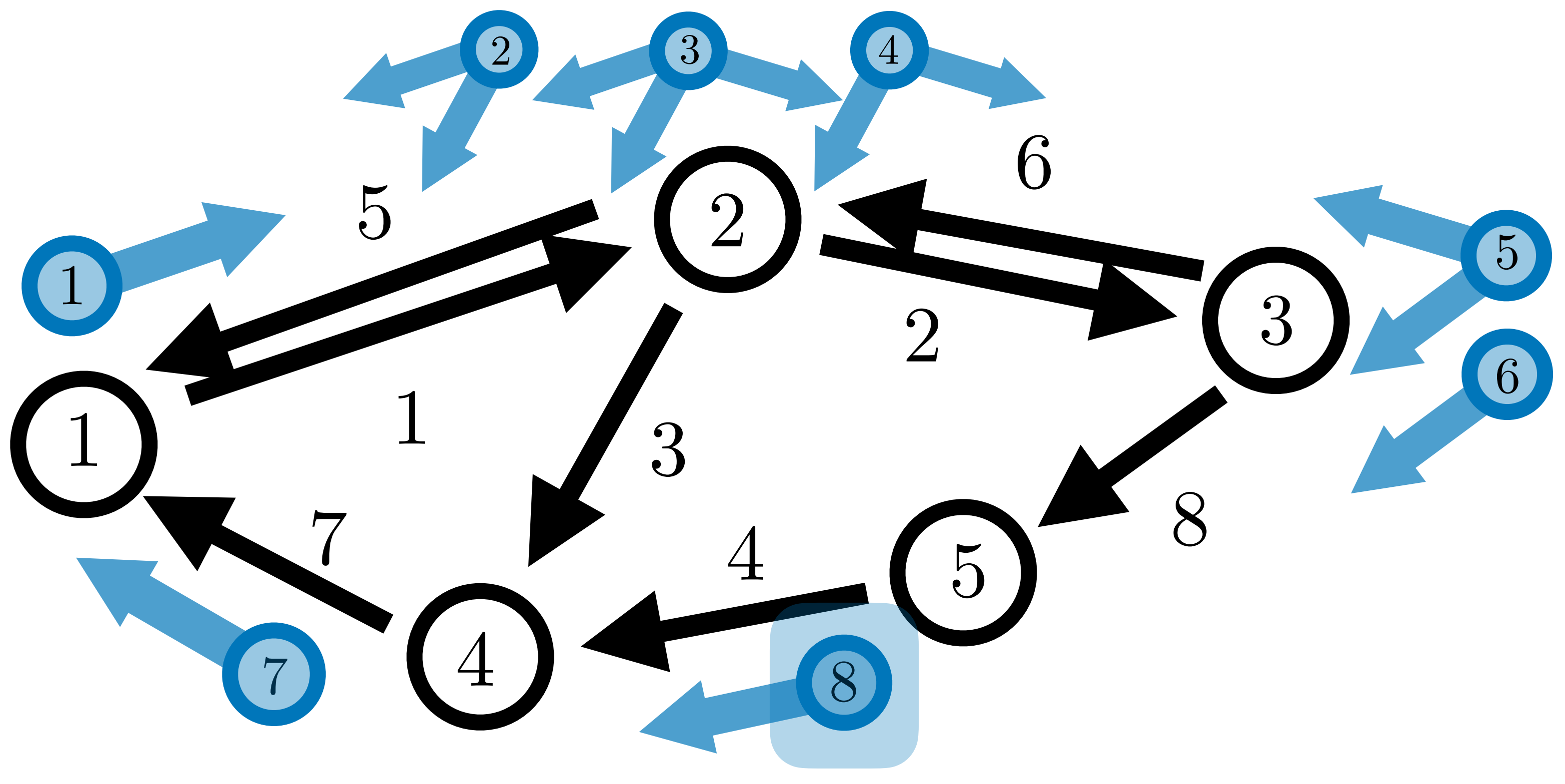

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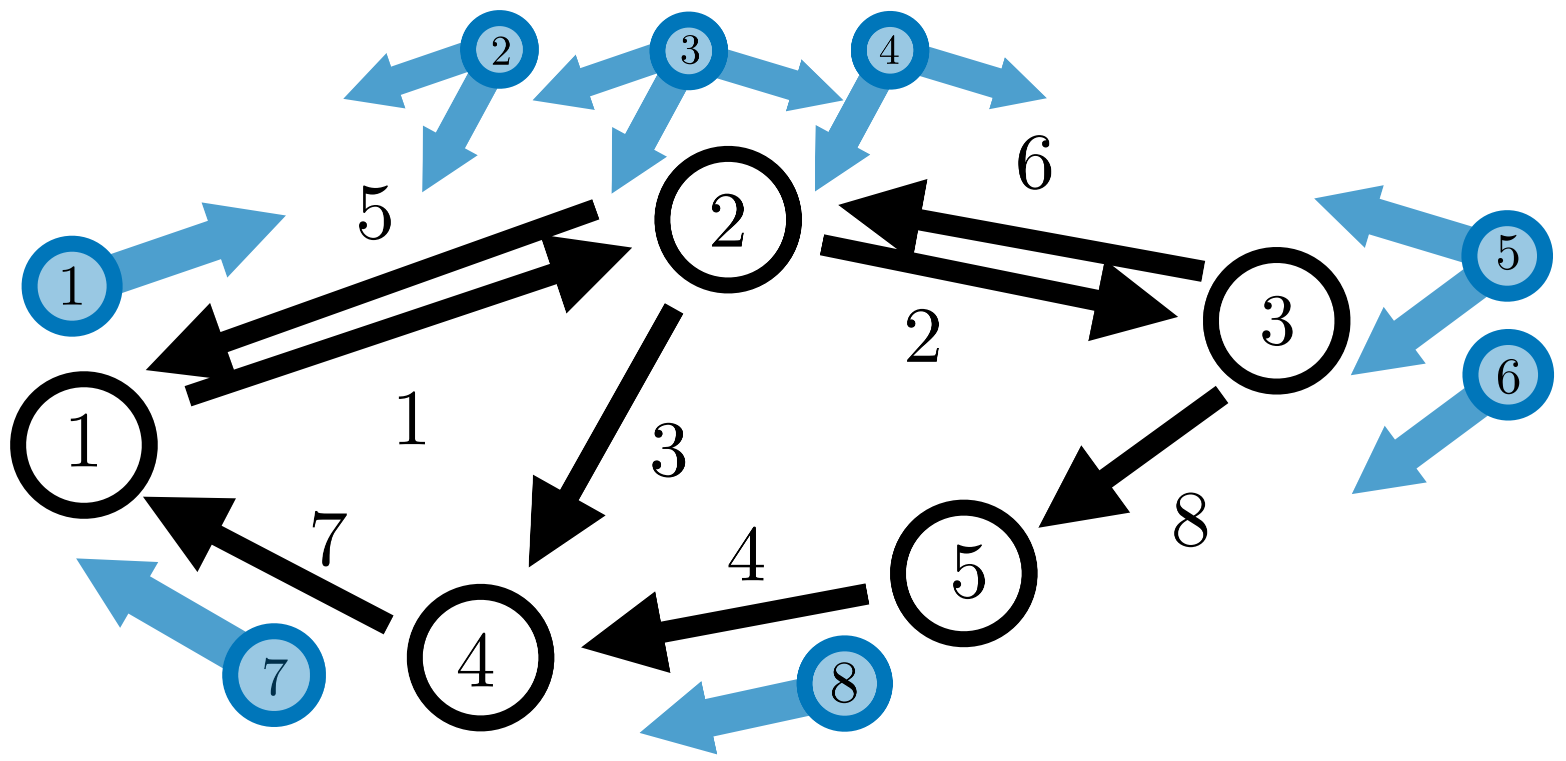
Transition Kernel



$$E_{\mathcal{A}} = E_{\text{out}} W$$
$$P = E_{\text{in}} W$$

Graph:

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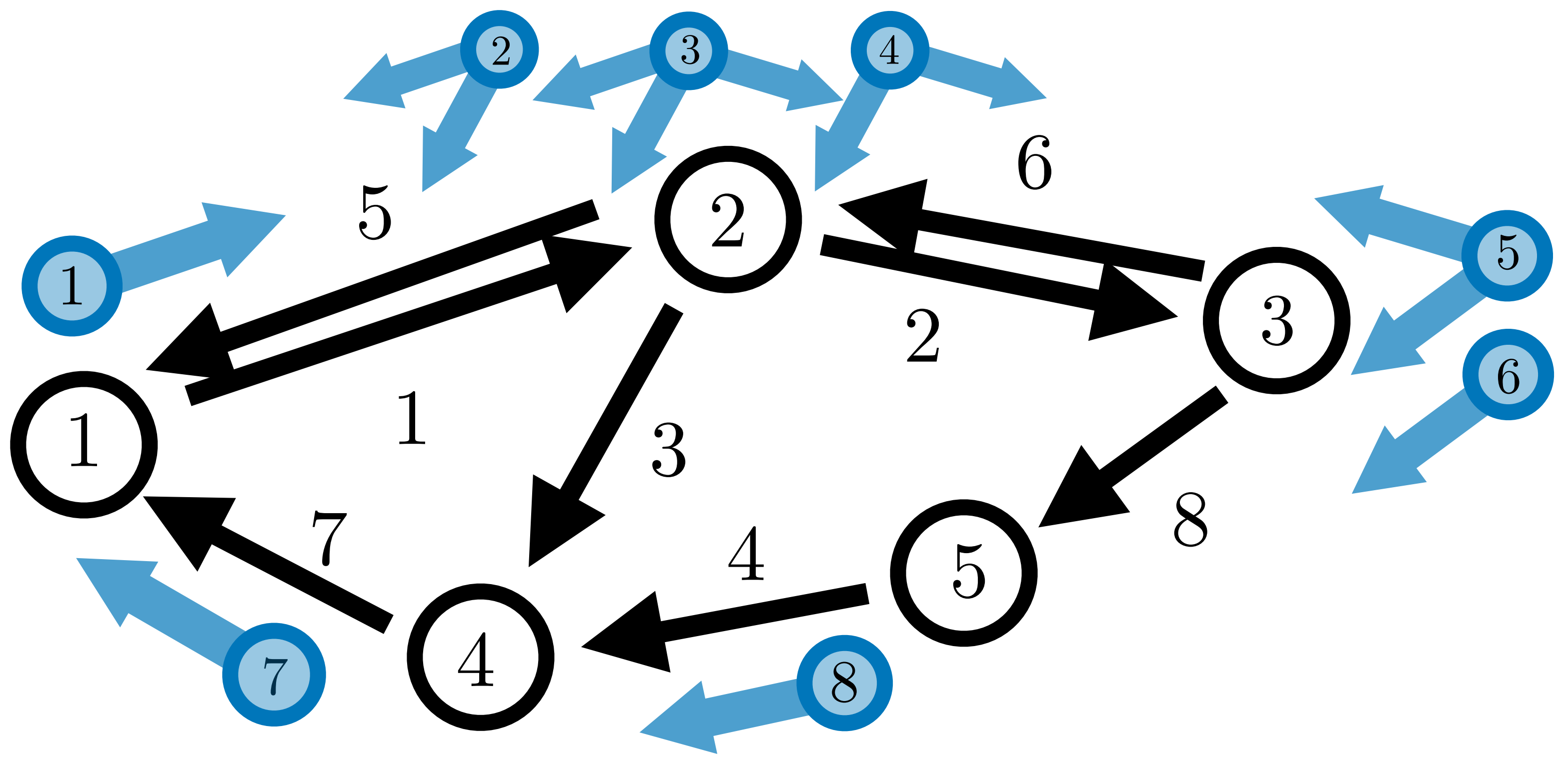

$$E_{\mathcal{A}} = E_{\text{out}} W$$

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Incidence Matrices

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Transition Kernel

Graph:

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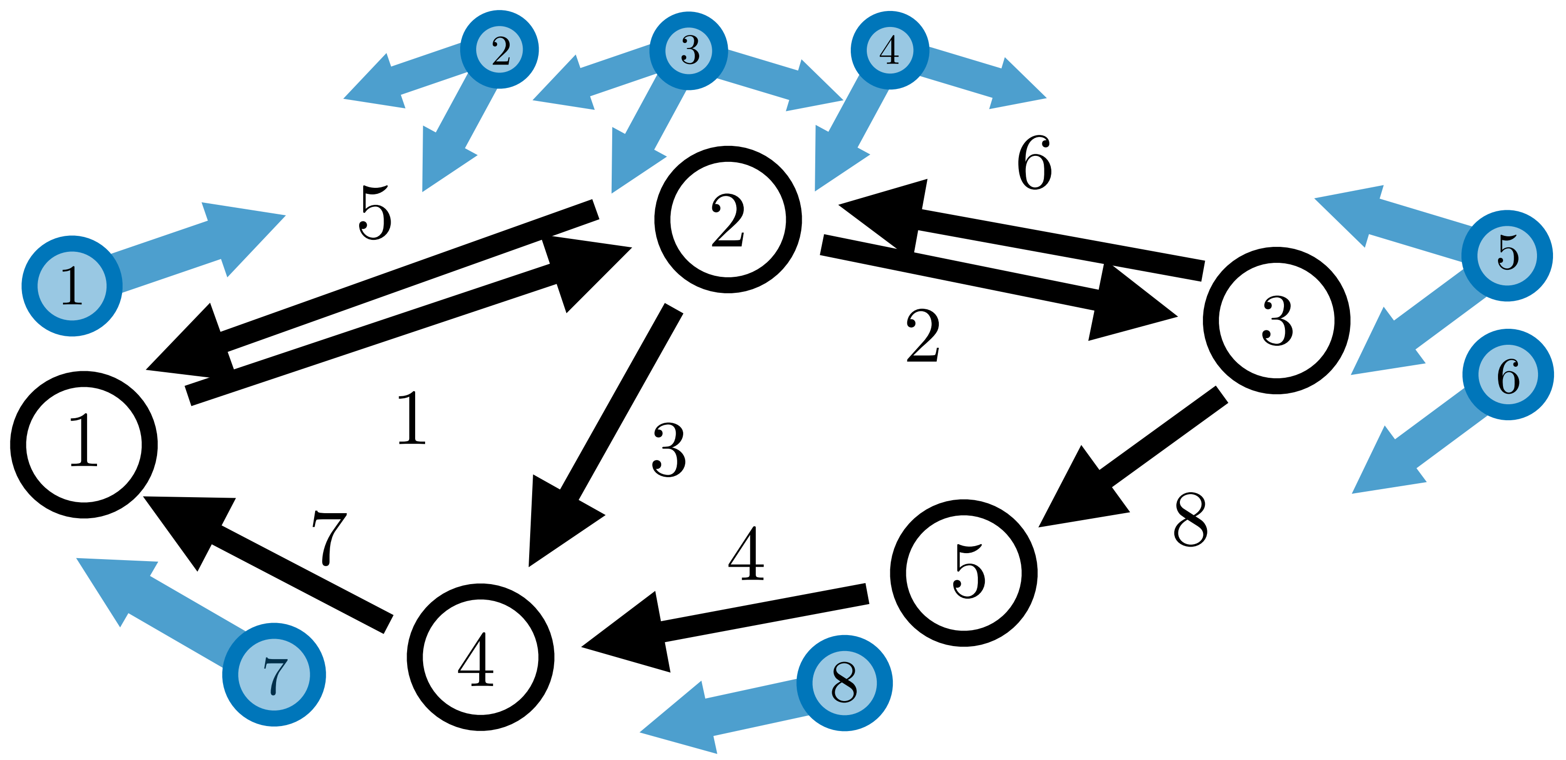

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Policy

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$$[\ E_{\text{out}}\ ]_{ve} = \begin{cases} 1; & \text{if } e \text{ out of } v \\ 0; & \text{otherwise} \end{cases}$$

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Transition Kernel

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$W \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{A}|}$

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$$[\ P\ ]_{s'a} = \begin{cases} \text{Prob}(s'|s,a); & \text{prob. of trans from } s \text{ to } s' \text{ given } a \\ 0; & \text{otherwise} \end{cases}$$

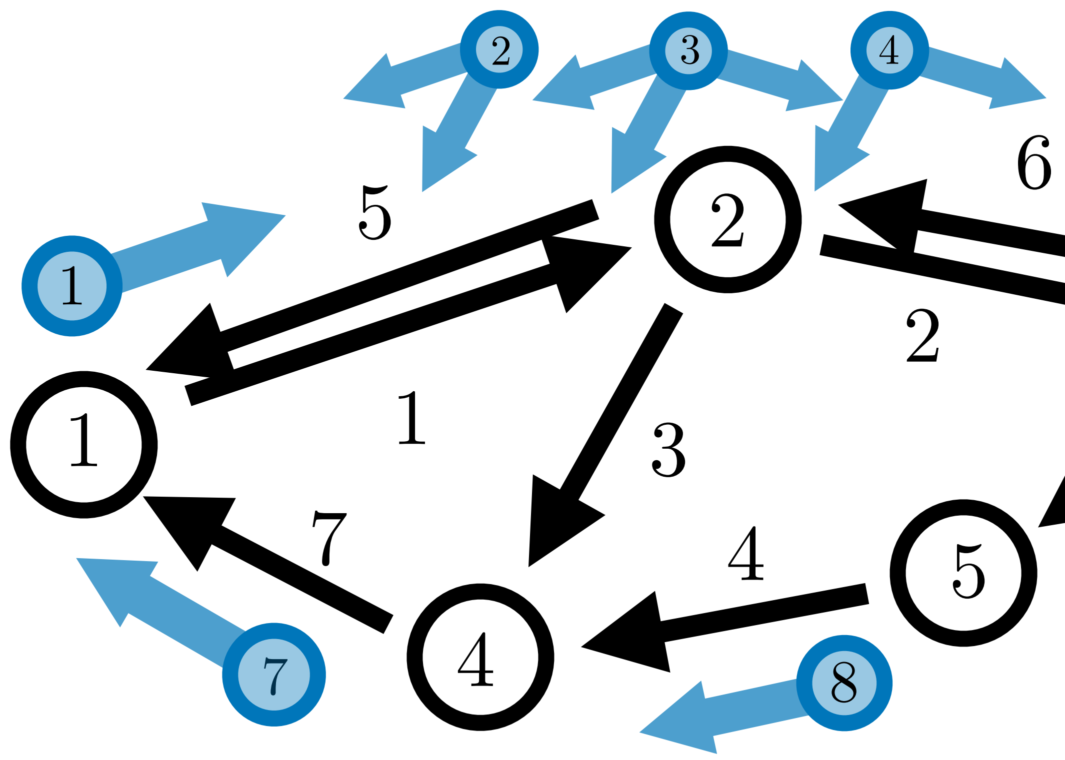
$$[\ W\ ]_{ea} = \begin{cases} \text{Prob}(e|s,a); & \text{prob. of taking edge } e \text{ from } s \text{ given } a \\ 0; & \text{otherwise} \end{cases}$$

Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{S}|}$

$M \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$

$$[\ \Pi\ ]_{as} = \begin{cases} \text{Prob}(a|s) & ;\ \text{prob. of taking } a \text{ given being in } s \\ 0 & \text{otherwise} \end{cases}$$



$$E_{\mathcal{A}} = E_{\text{out}}W$$

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$$M = P\Pi$$

$$I = E_{\mathcal{A}}\Pi$$

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$$P = \begin{bmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix}$$

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Policy

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

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Incidence Matrices

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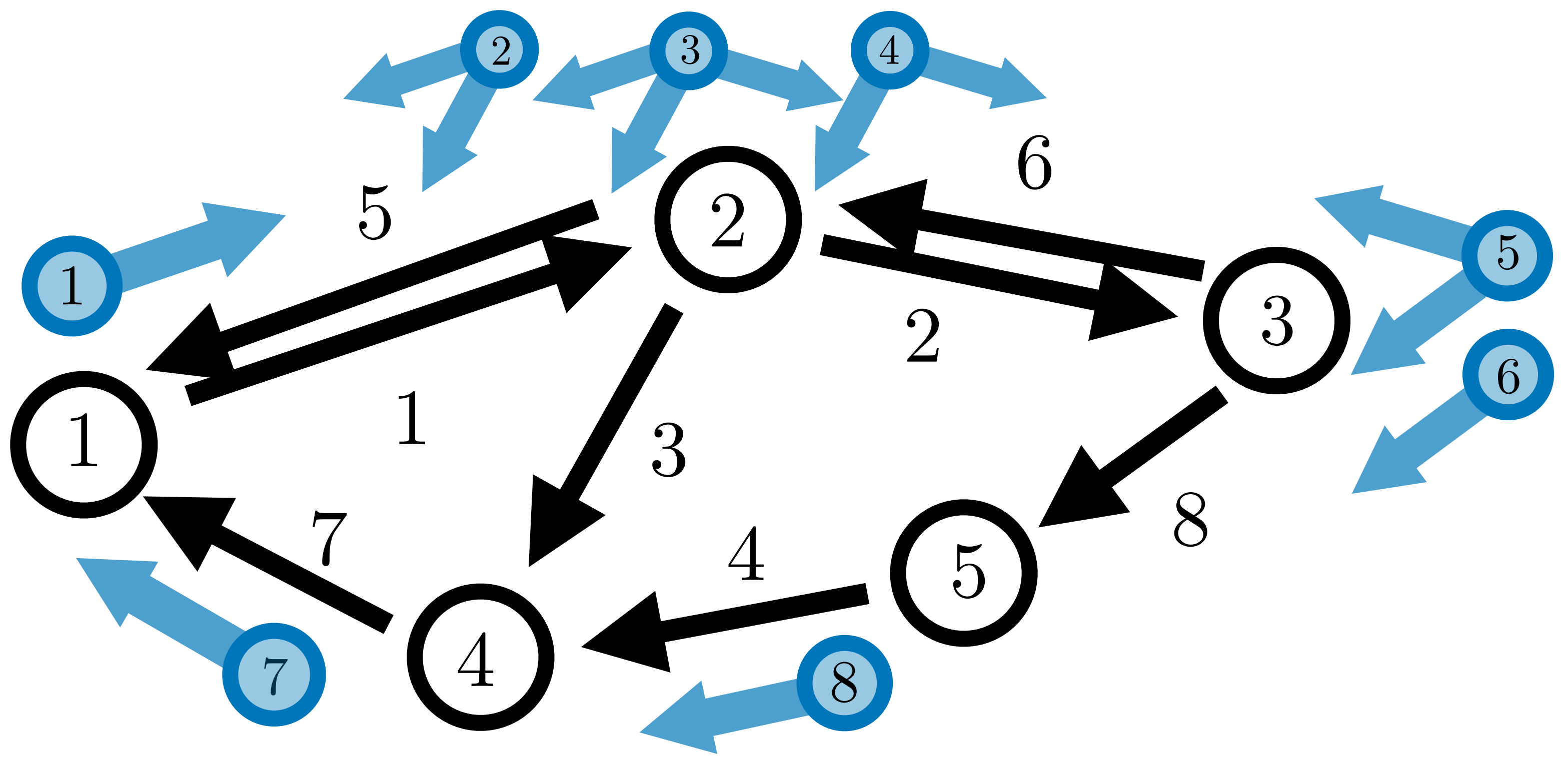

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Policy

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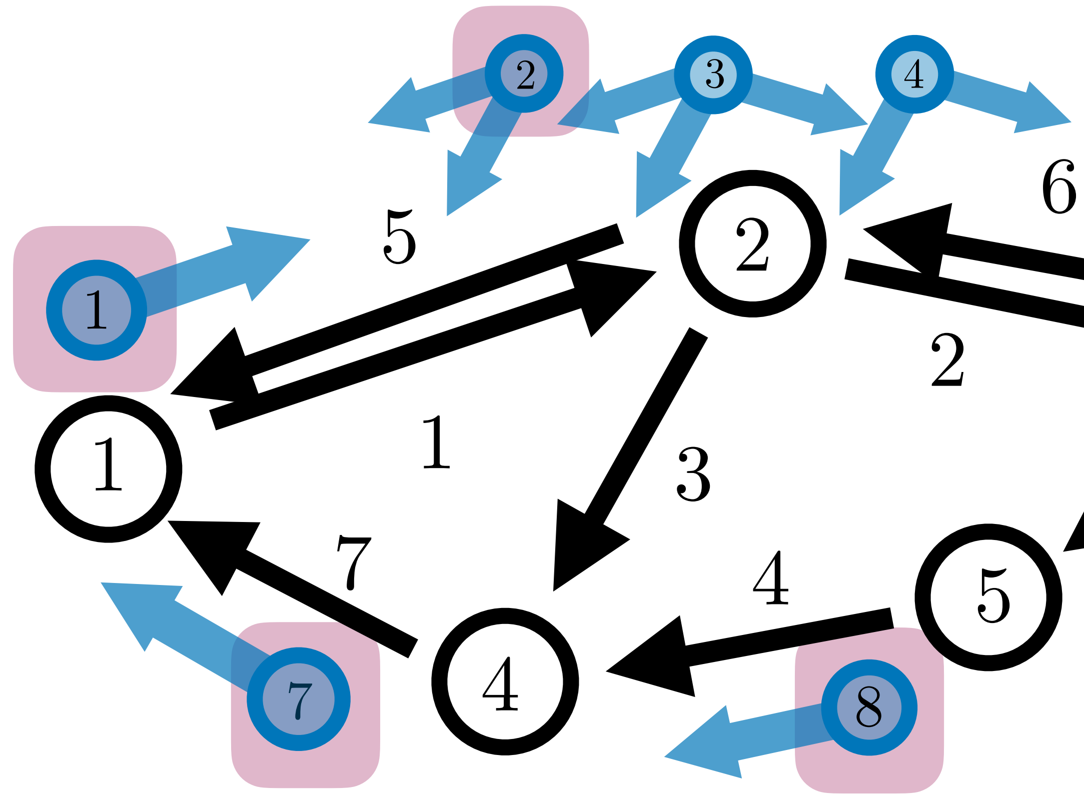
actions

probability of actions conditioned on state

Policy



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Policy

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Policy

states

actions

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probability of actions conditioned on state

“selecting actions…”

Policy

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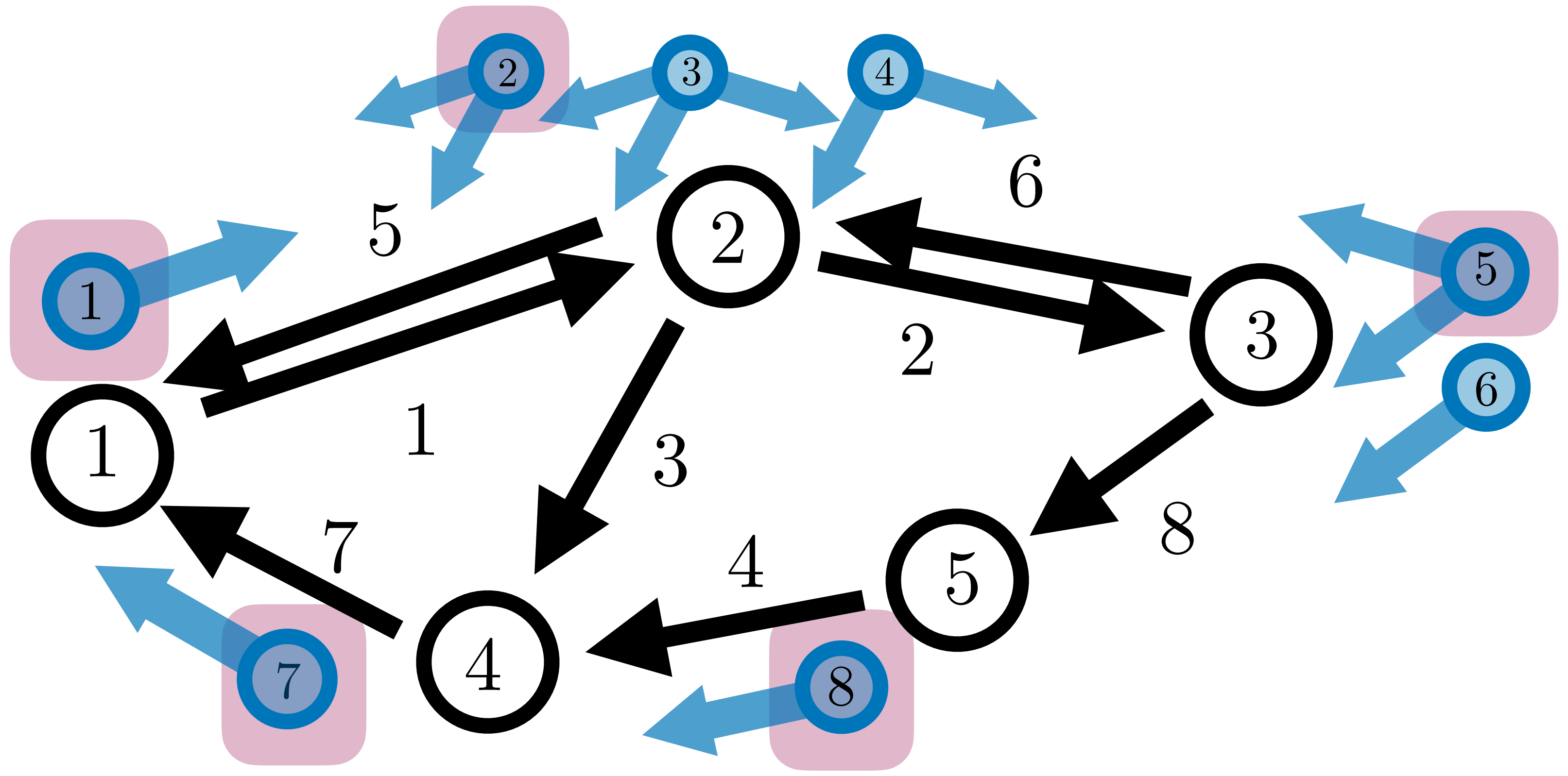

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states

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Policy

actions

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Markov Matrix

$$M = P\Pi = \begin{bmatrix} 0 & 0.5 & 0 & 1 & 0 \\ 1 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0 \end{bmatrix}$$

“selecting actions…”

Policy

Graph:

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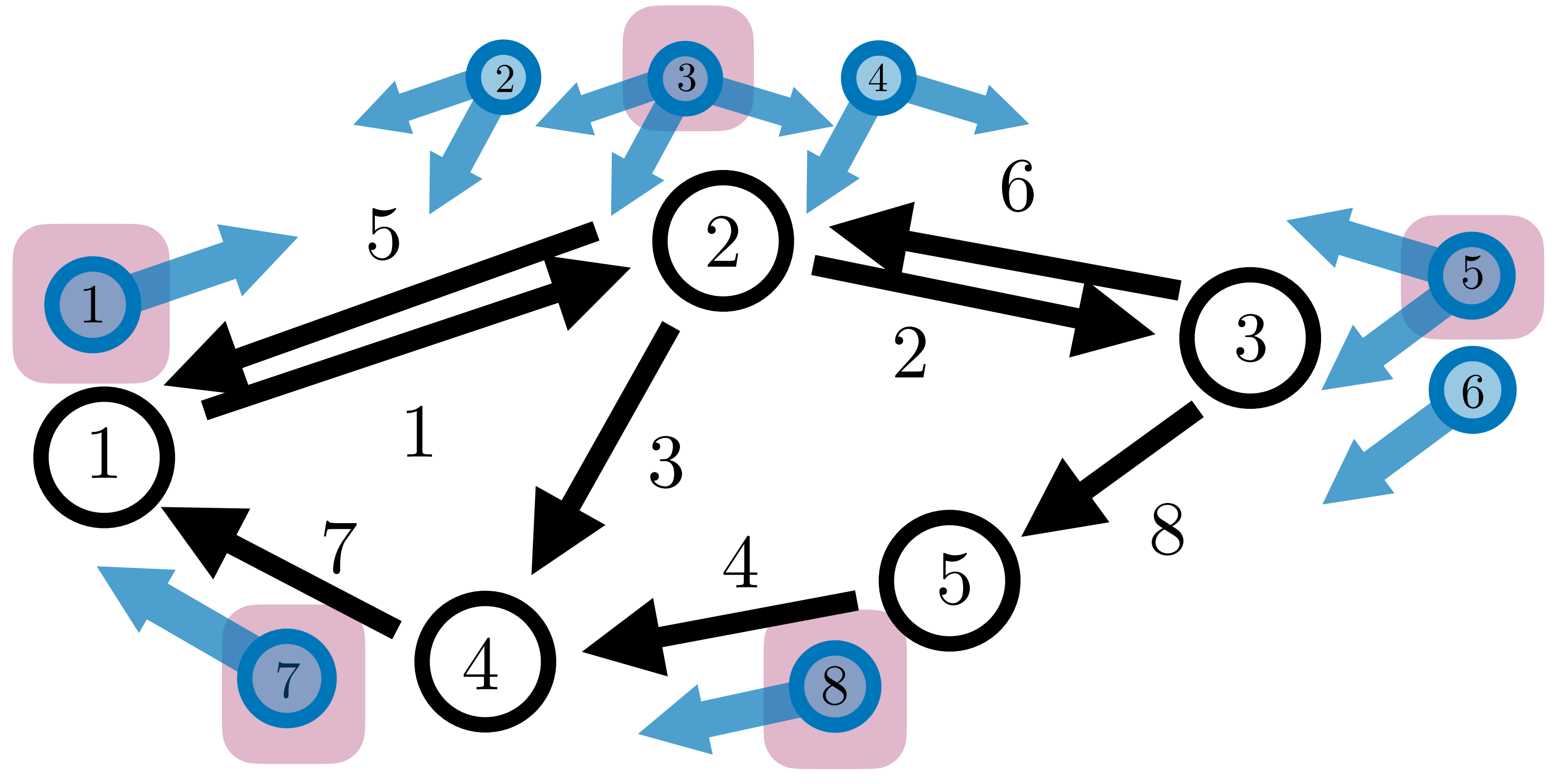

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states

Policy

actions

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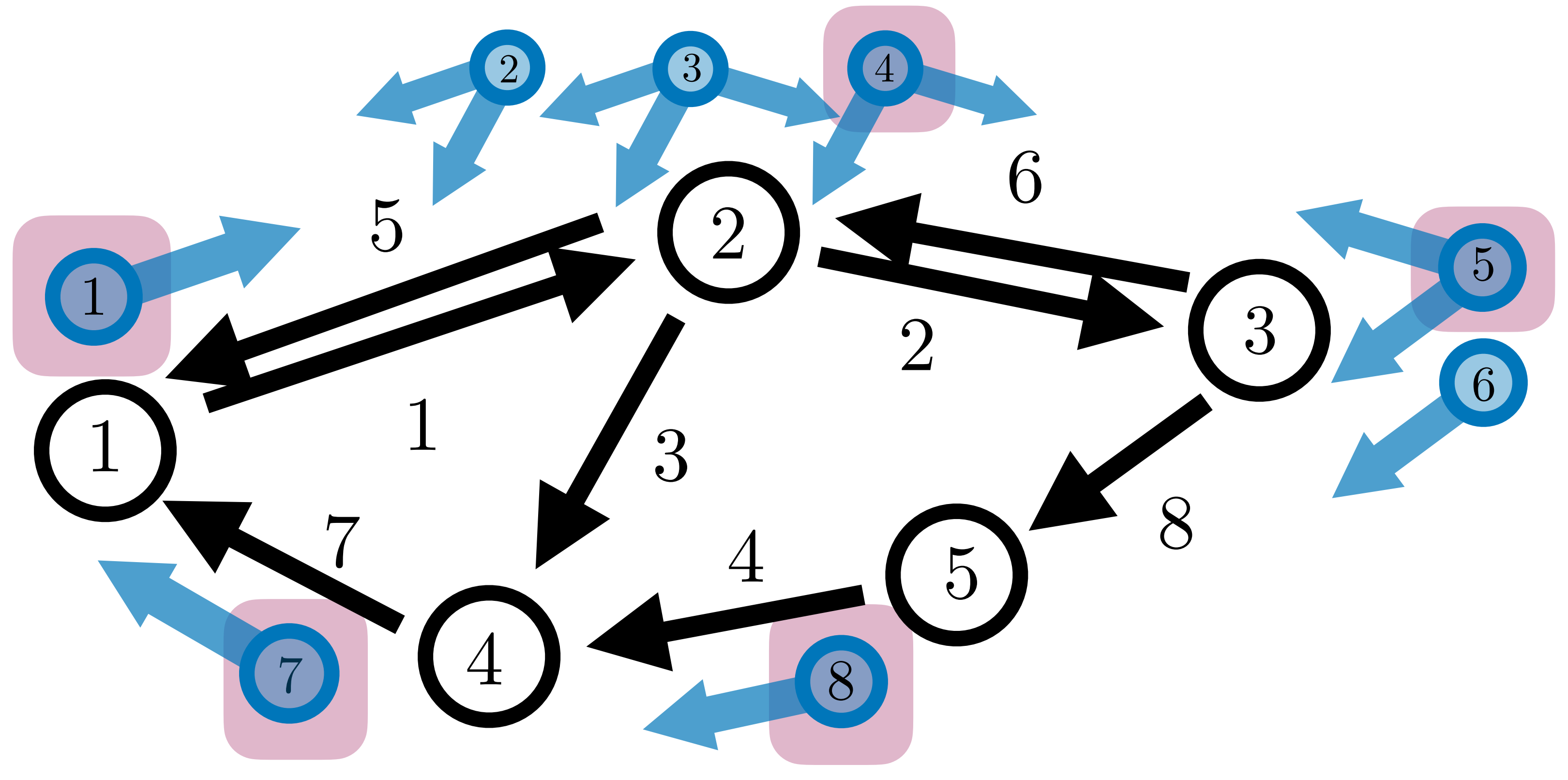

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$$P = \begin{bmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.5 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix}$$

states

Policy

Markov Matrix

actions

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = P\Pi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0 \end{bmatrix}$$

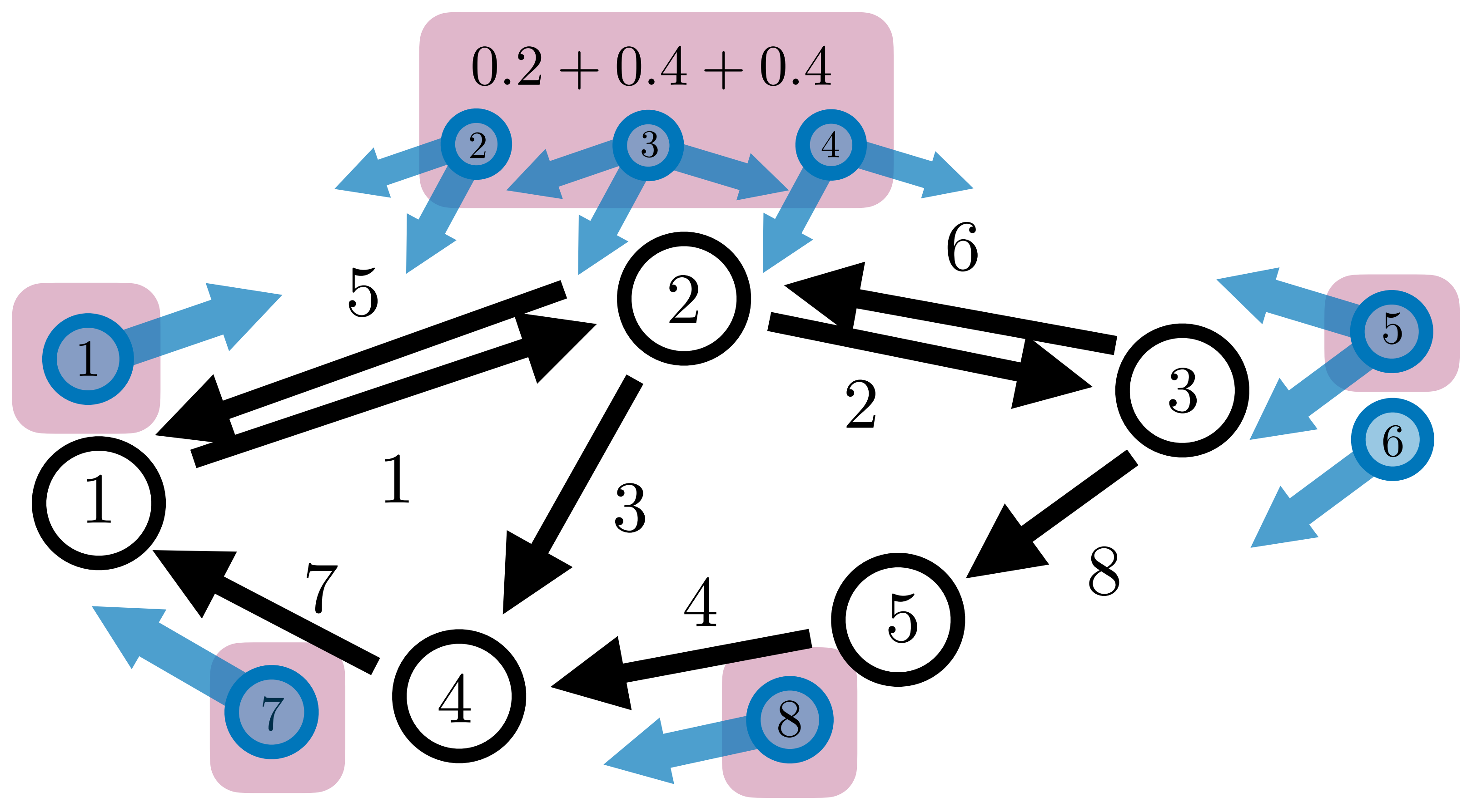
Policy

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

Edges $e \in \mathcal{E}$ $e = (v, v')$



$$E_{\mathcal{A}} = E_{\text{out}} W$$
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Incidence Matrices

$E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $E = E_{\text{in}} - E_{\text{out}}$

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$W \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{A}|}$

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Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{S}|}$

$M \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{S}|}$

$$[\, \Pi \,]_{as} = \begin{cases} \text{Prob}(a|s) & ; \text{ prob. of taking } a \text{ given being in } s \\ 0 & \text{otherwise} \end{cases}$$

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Mass conservation with source-sink

$$[\, E_{\mathcal{A}} - P \,] x = [\, E_{\text{out}} - E_{\text{in}} \,] W x = S$$

$$E_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix}$$

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Policy

$$\Pi = \begin{array}{cc} & \text{states} \\ \text{actions} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Markov Matrix

$$M = P\Pi = \begin{bmatrix} 0 & 0.22 & 0 & 1 & 0 \\ 1 & 0 & 0.5 & 0 & 0 \\ 0 & 0.32 & 0 & 0 & 0 \\ 0 & 0.46 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 & 0 \end{bmatrix}$$

Policy

Graph:

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$$[\,W\,]_{ea} = \begin{cases} \text{Prob}(e|s,a); & \text{prob. of taking edge } e \text{ from } s \text{ given } a \\ 0; & \text{otherwise} \end{cases}$$

Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{S}|}$

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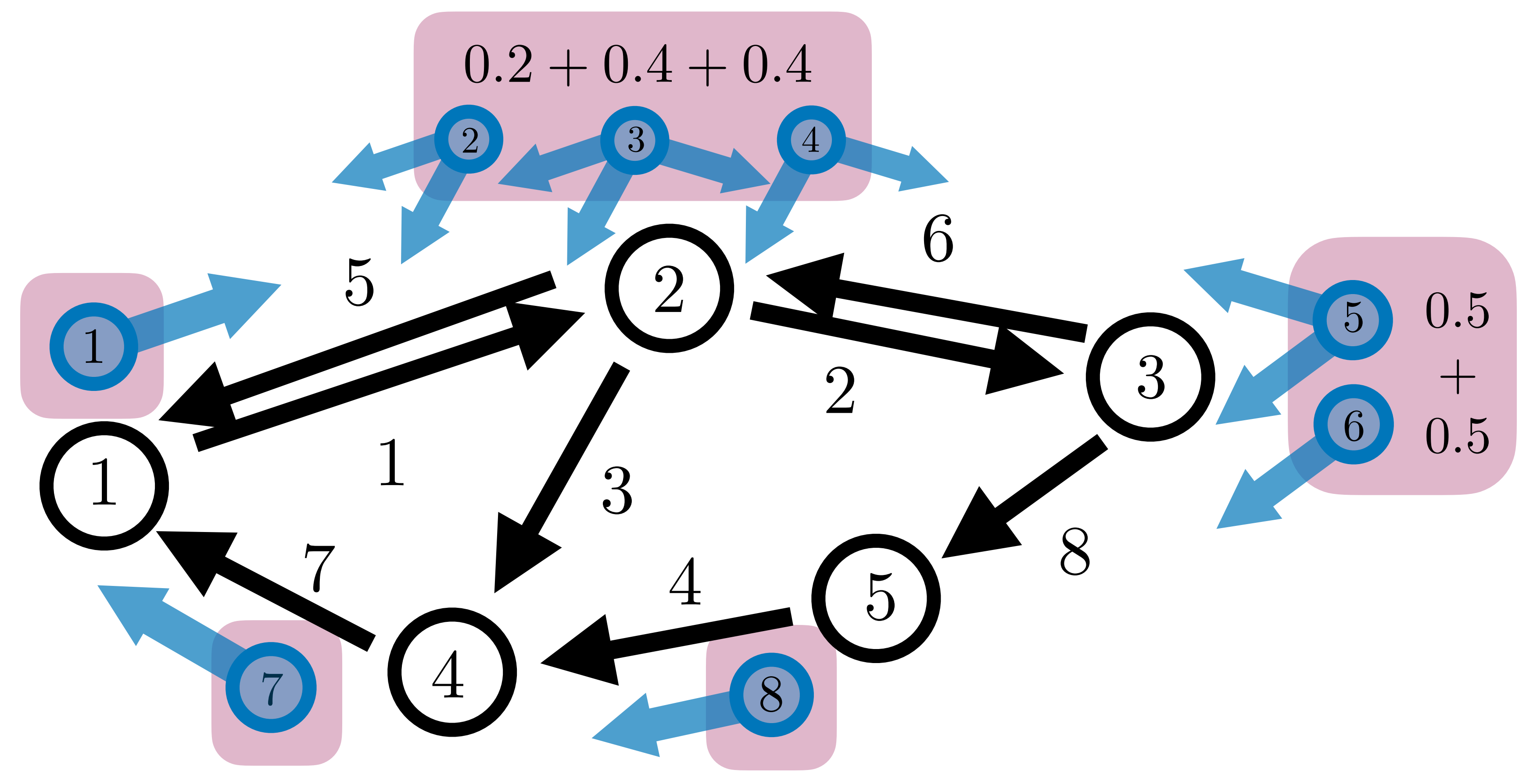

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states

Policy

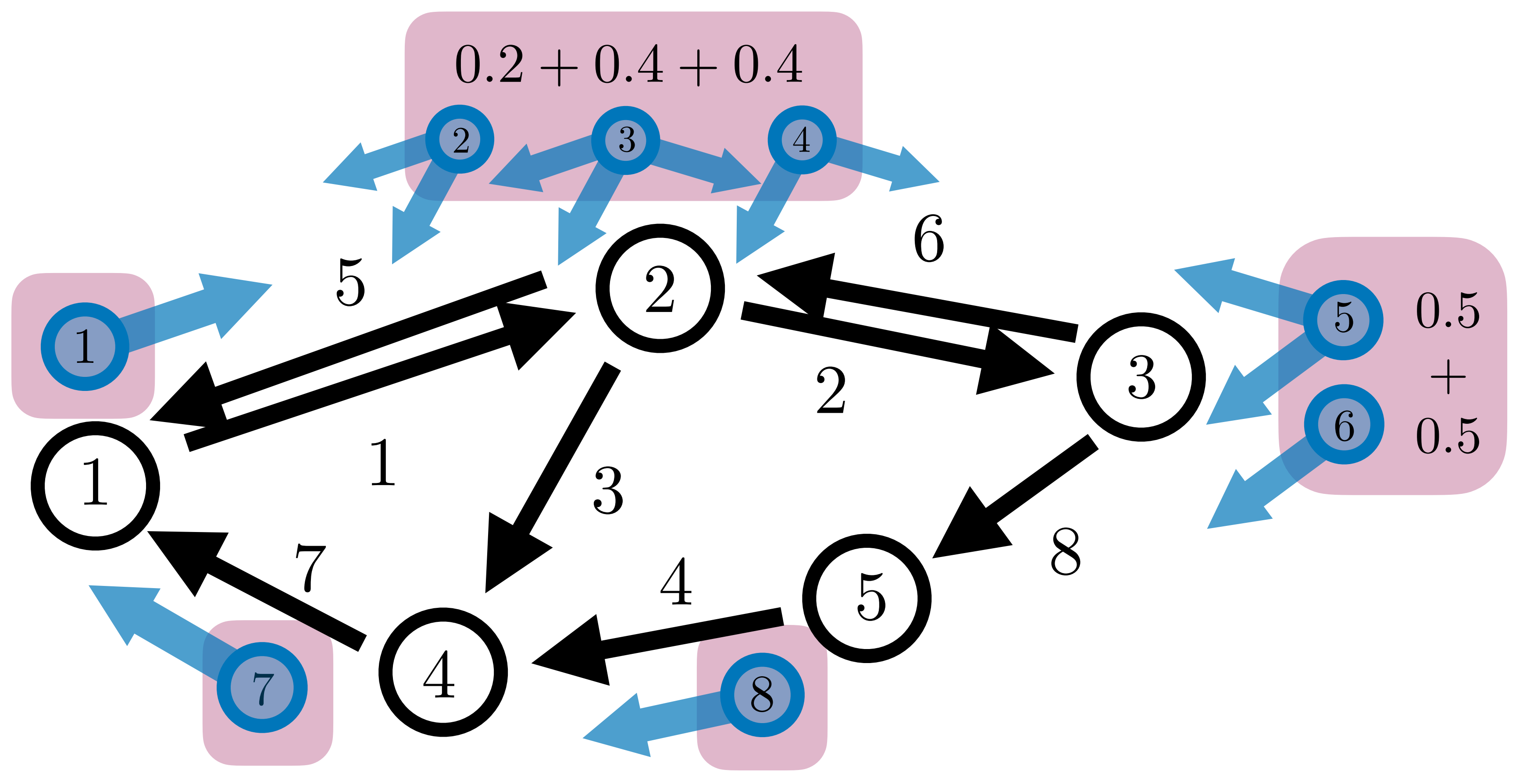
actions

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Markov Matrix

$$M = P\Pi = \begin{bmatrix} 0 & 0.22 & 0 & 1 & 0 \\ 1 & 0 & 0.25 & 0 & 0 \\ 0 & 0.32 & 0 & 0 & 0 \\ 0 & 0.46 & 0 & 0 & 1 \\ 0 & 0 & 0.75 & 0 & 0 \end{bmatrix}$$

Policy



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Graph:

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$$E_{\mathcal{A}} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \quad P = \begin{bmatrix} 0&0.5&0.3&0&0&0&1&0 \\ 1&0&0&0&0.5&0&0&0 \\ 0&0&0.3&0.5&0&0&0&0 \\ 0&0.5&0.4&0.5&0&0&0&1 \\ 0&0&0&0&0.5&1&0&0 \end{bmatrix} \quad W = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0.3&0.5&0&0&0&0 \\ 0&0.5&0.4&0.5&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0.5&0.3&0&0&0&0&0 \\ 0&0&0&0&0.5&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0.5&1&0&0 \end{bmatrix}$$

Transition Kernel

$P \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{A}|}$

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Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{S}|}$

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Policy

Markov Matrix

states

actions

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also…

$$N = \Pi P \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$$

Policy

Graph:

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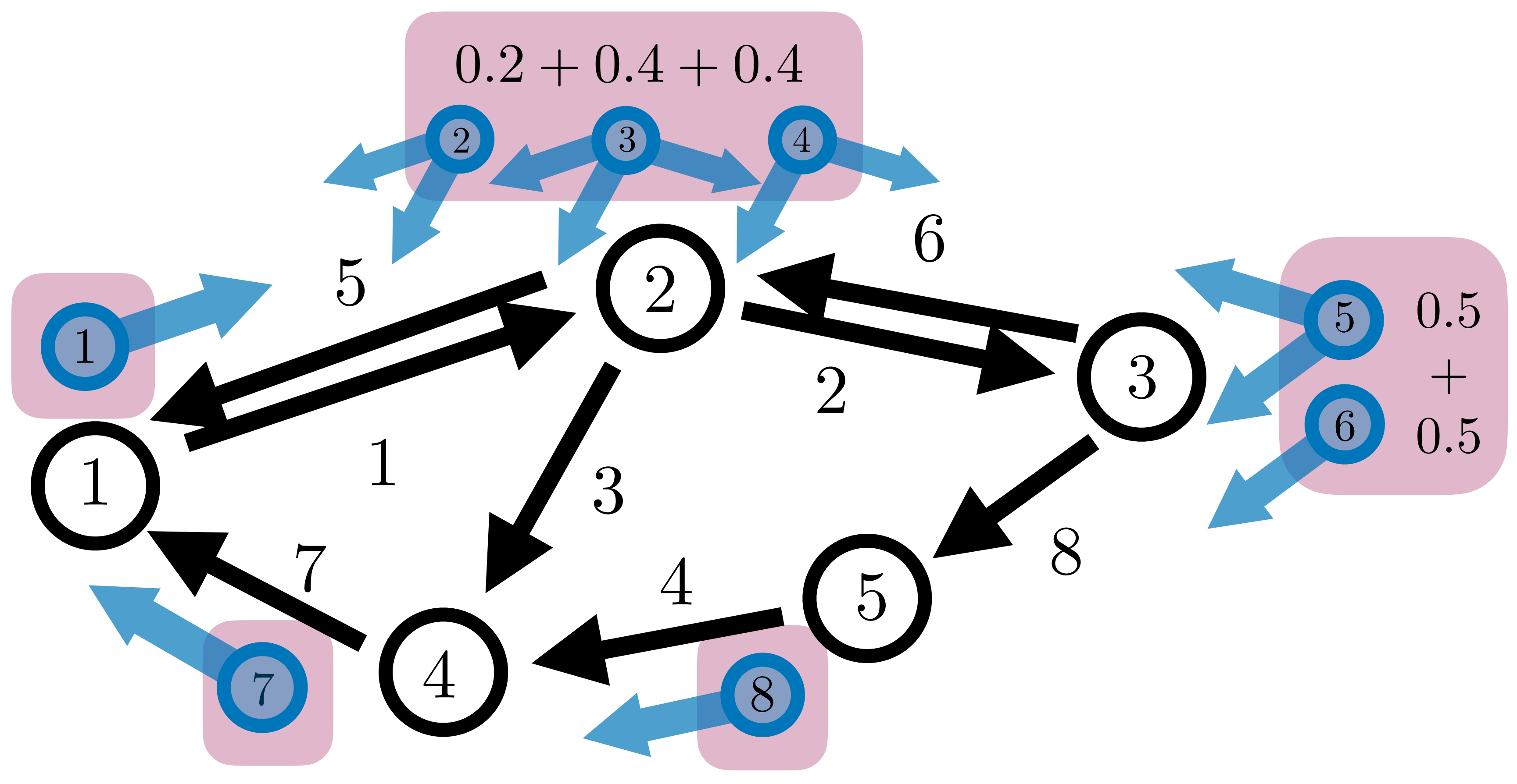

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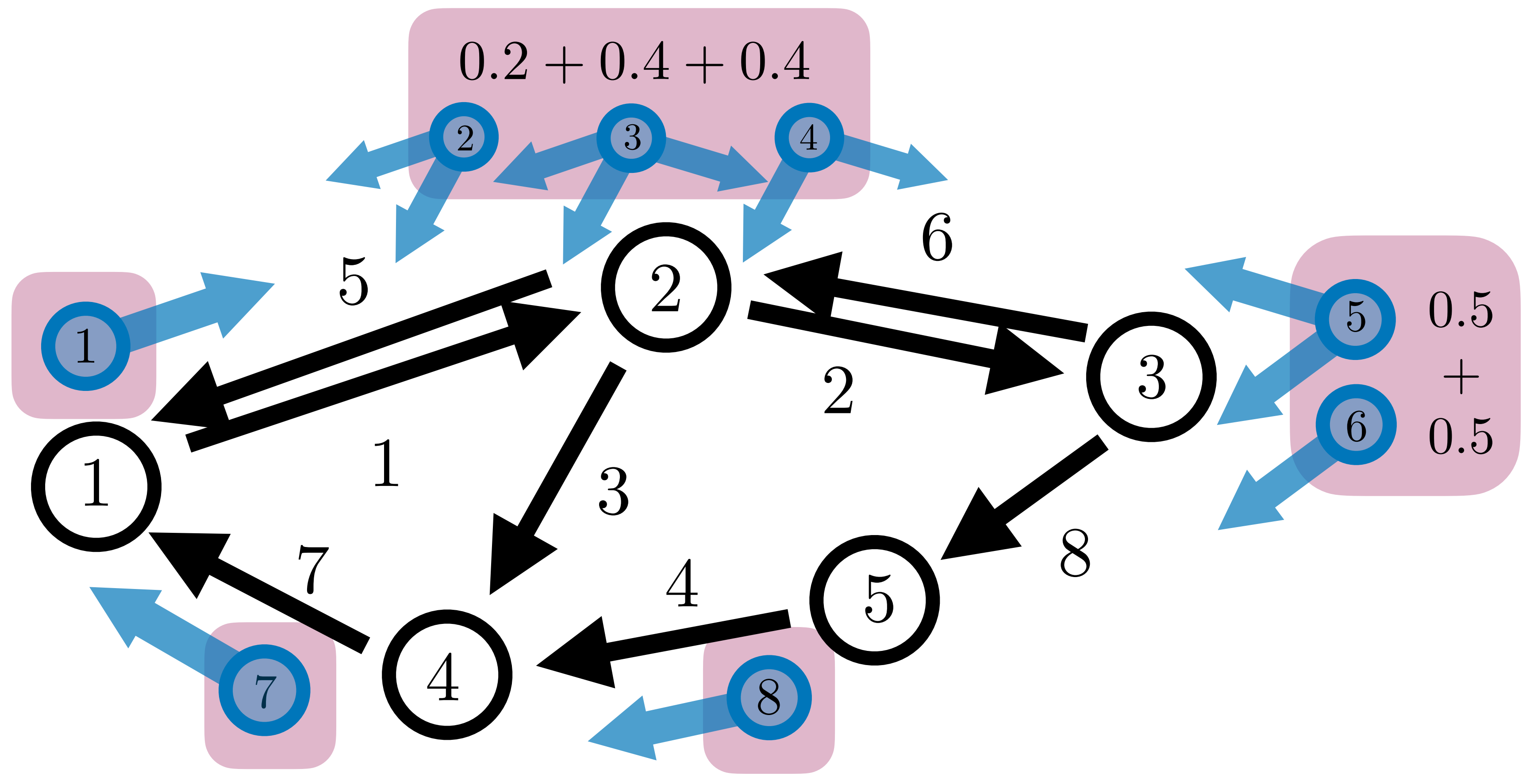

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$$E_{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0.5 & 0.3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.4 & 0.5 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 0 & 0 \end{bmatrix}$$

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$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0.22 & 0 & 1 & 0 \\ 1 & 0 & 0.25 & 0 & 0 \\ 0 & 0.32 & 0 & 0 & 0 \\ 0 & 0.46 & 0 & 0 & 1 \\ 0 & 0 & 0.75 & 0 & 0 \end{bmatrix}$$

Column Stochastic

positive & sum to 1

$\mathbf{1}^T E_{\text{out}} = \mathbf{1}^T$ $\mathbf{1}^T E_{\mathcal{A}} = \mathbf{1}^T$ $\mathbf{1}^T W = \mathbf{1}^T$ $\mathbf{1}^T \Pi = \mathbf{1}^T$

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Transition Kernel

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

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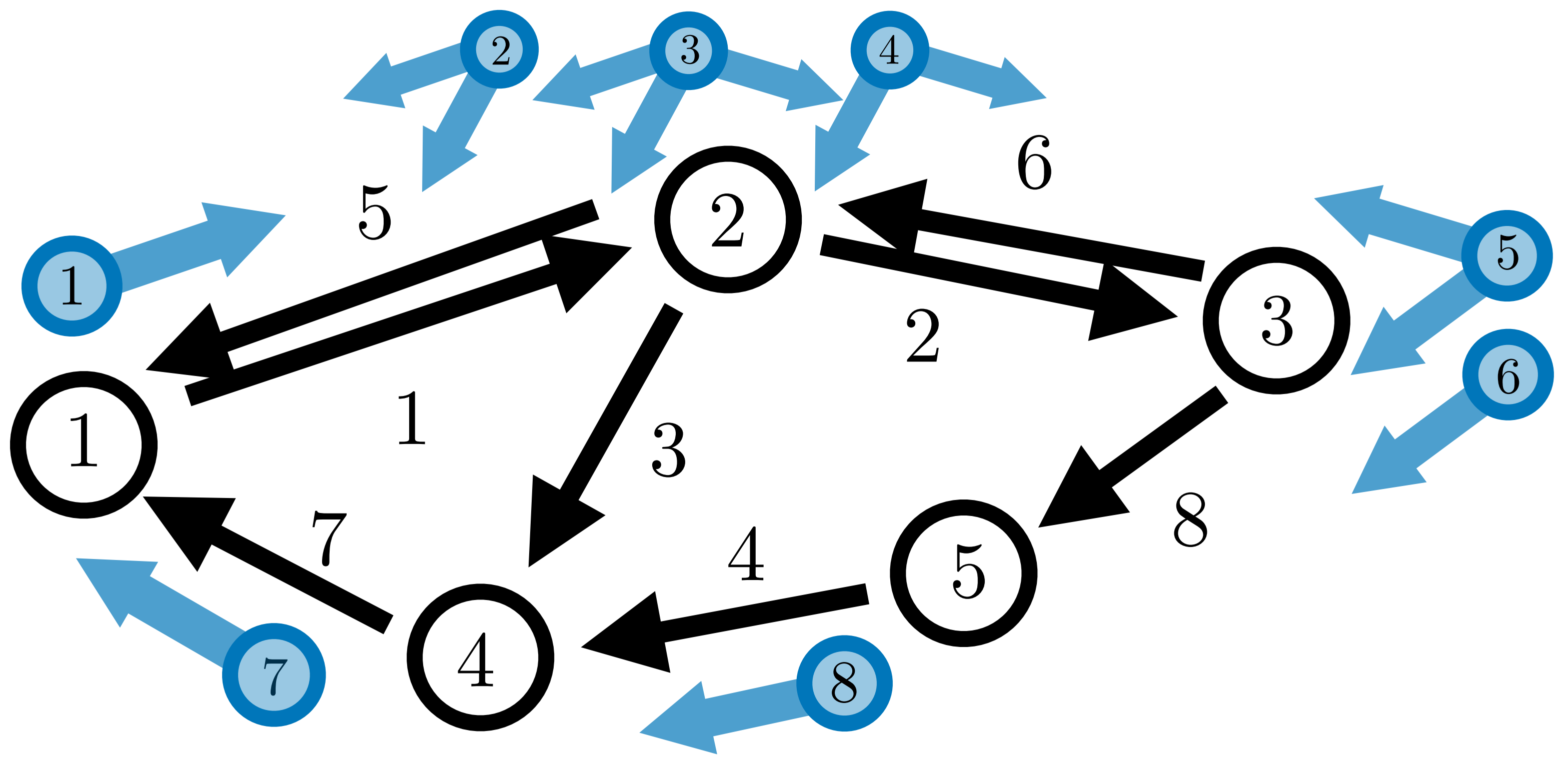

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Transition Kernel Laplacian

…specific weighted graph Laplacian

$$L_P = (E_{\mathcal{A}} - P)\,(E_{\mathcal{A}} - P)^T = E W W^T E^T$$

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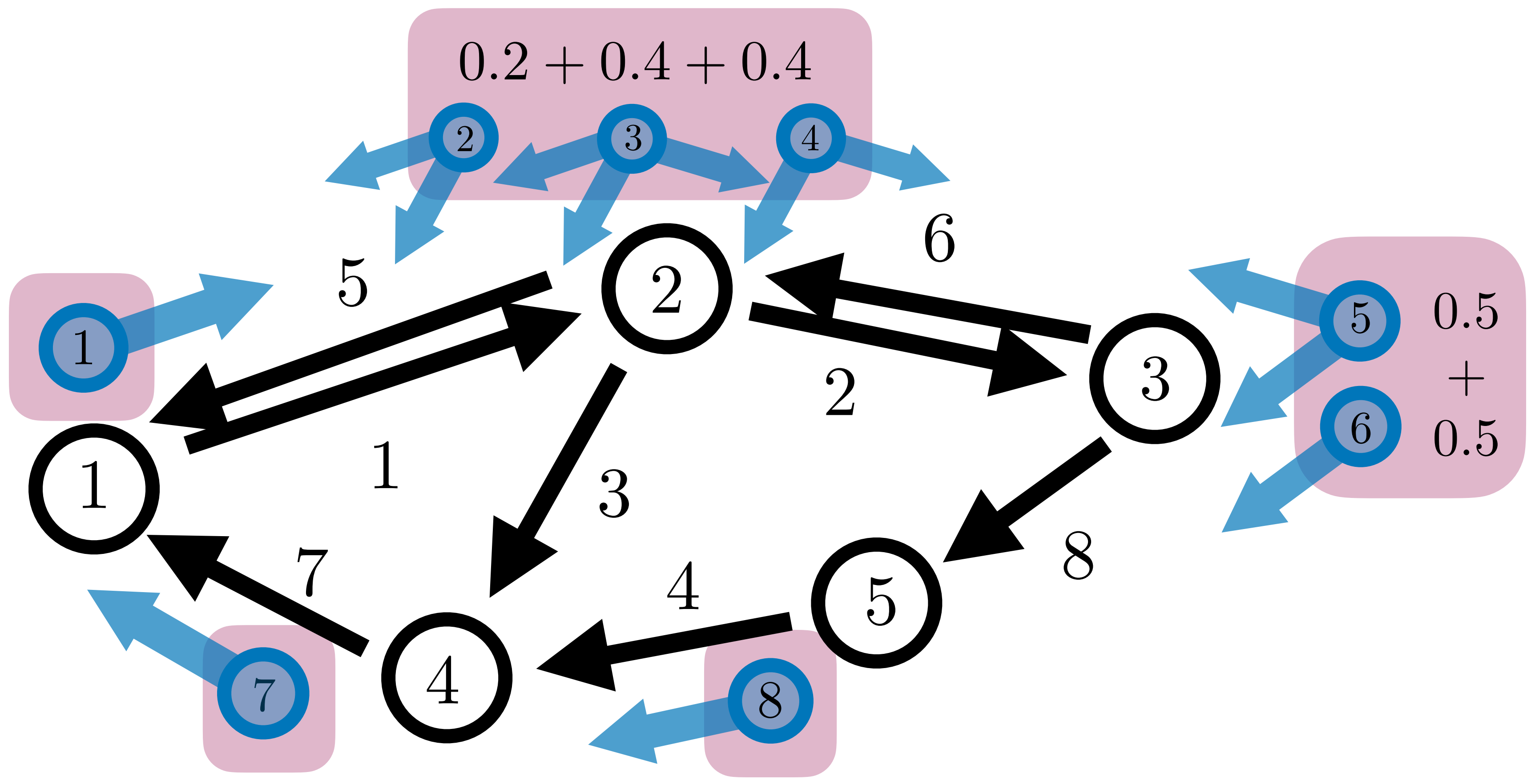

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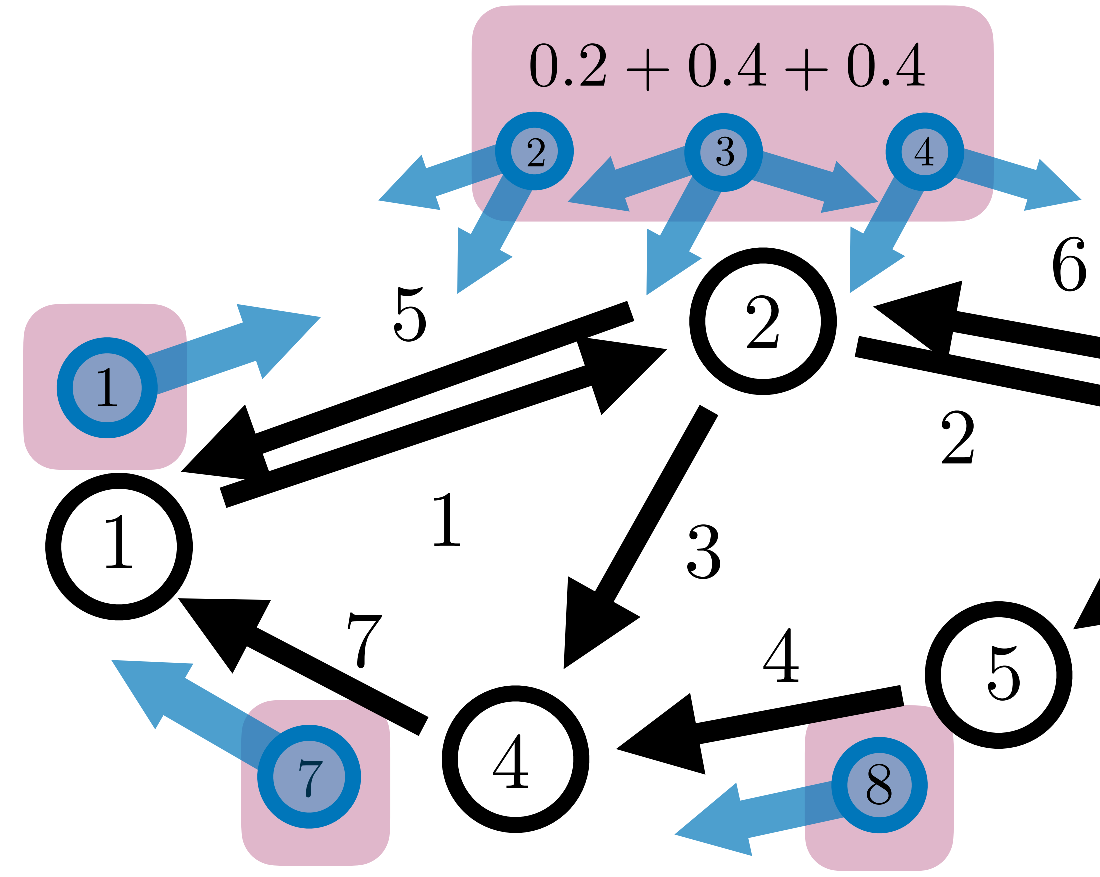

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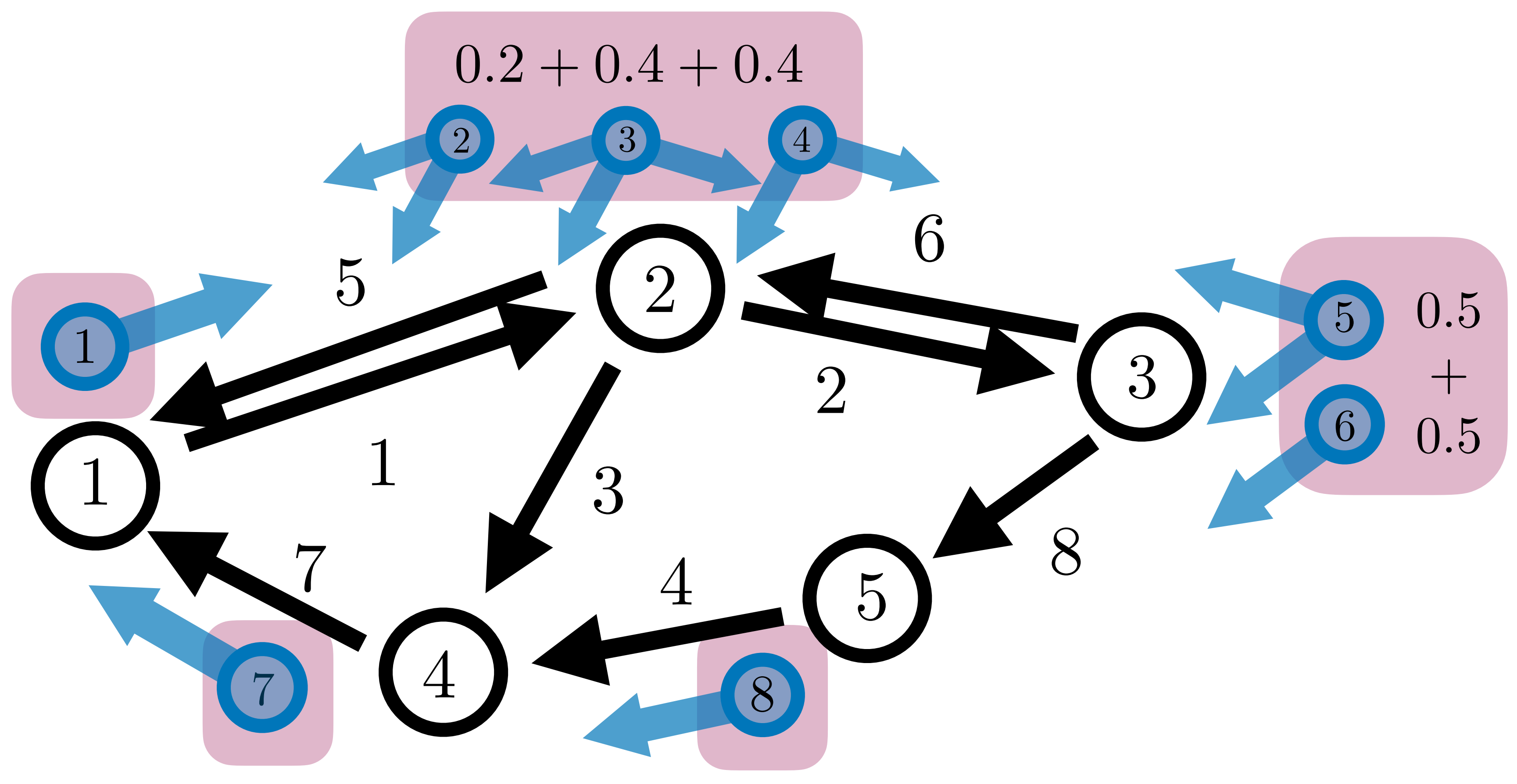

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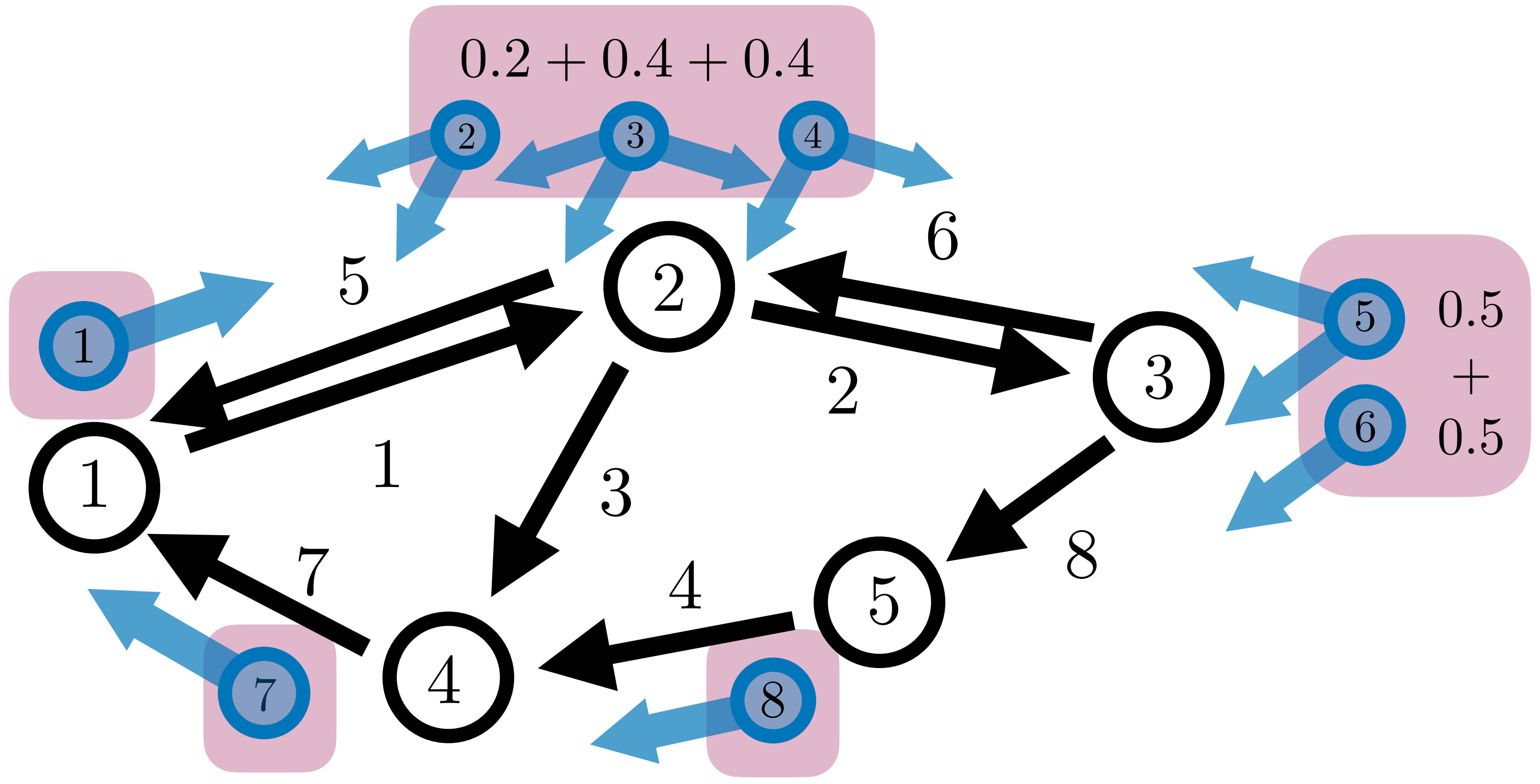

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Steady-state (state-action) distribution

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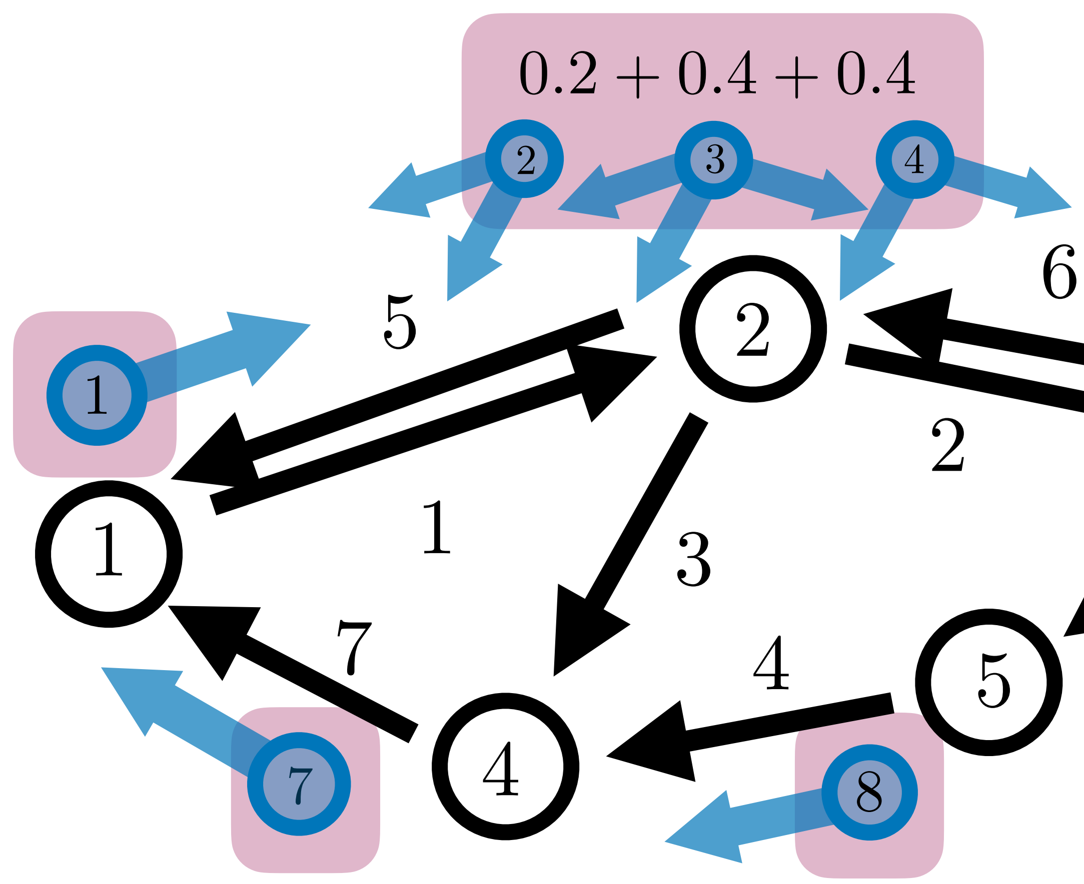

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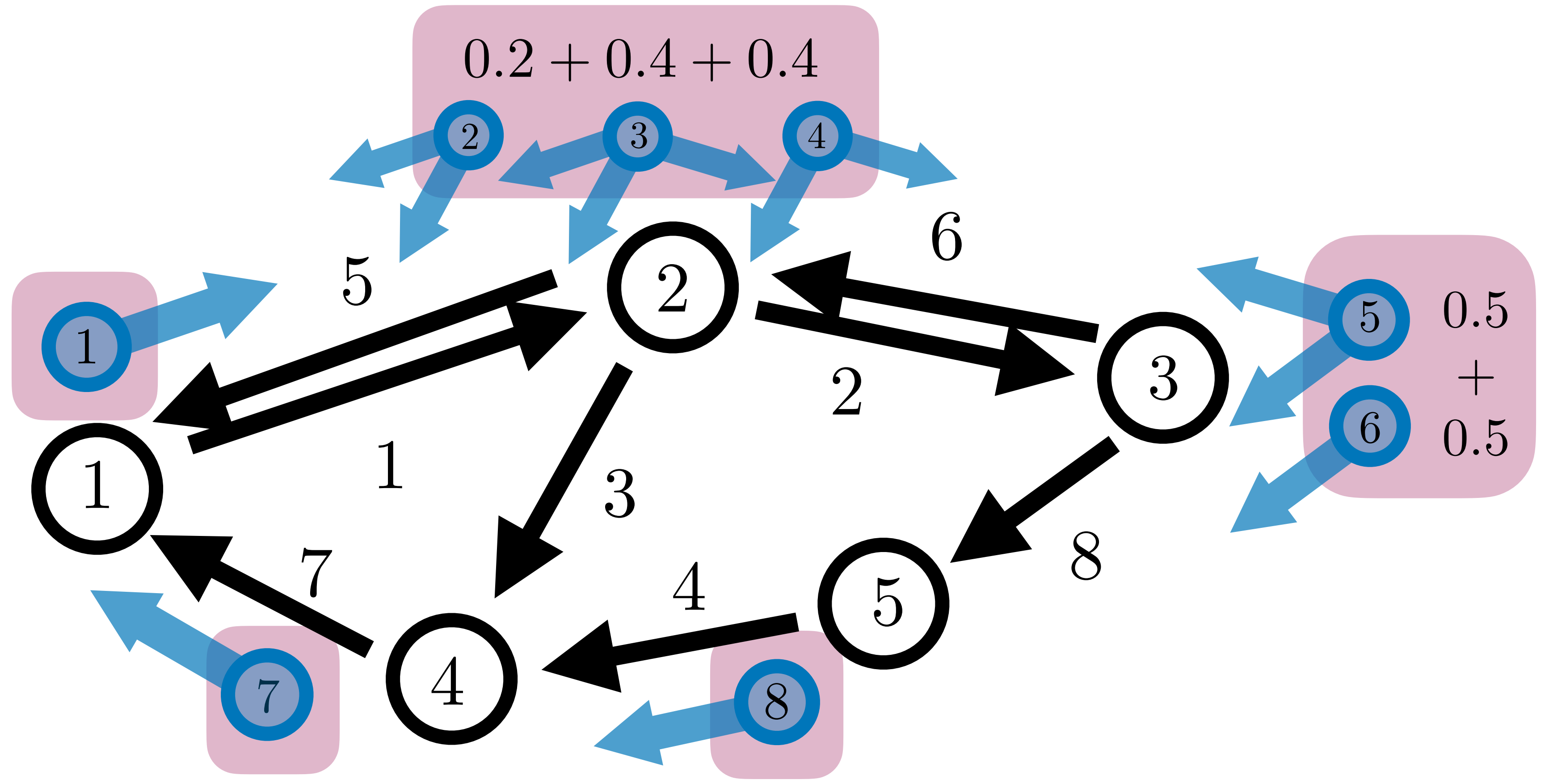

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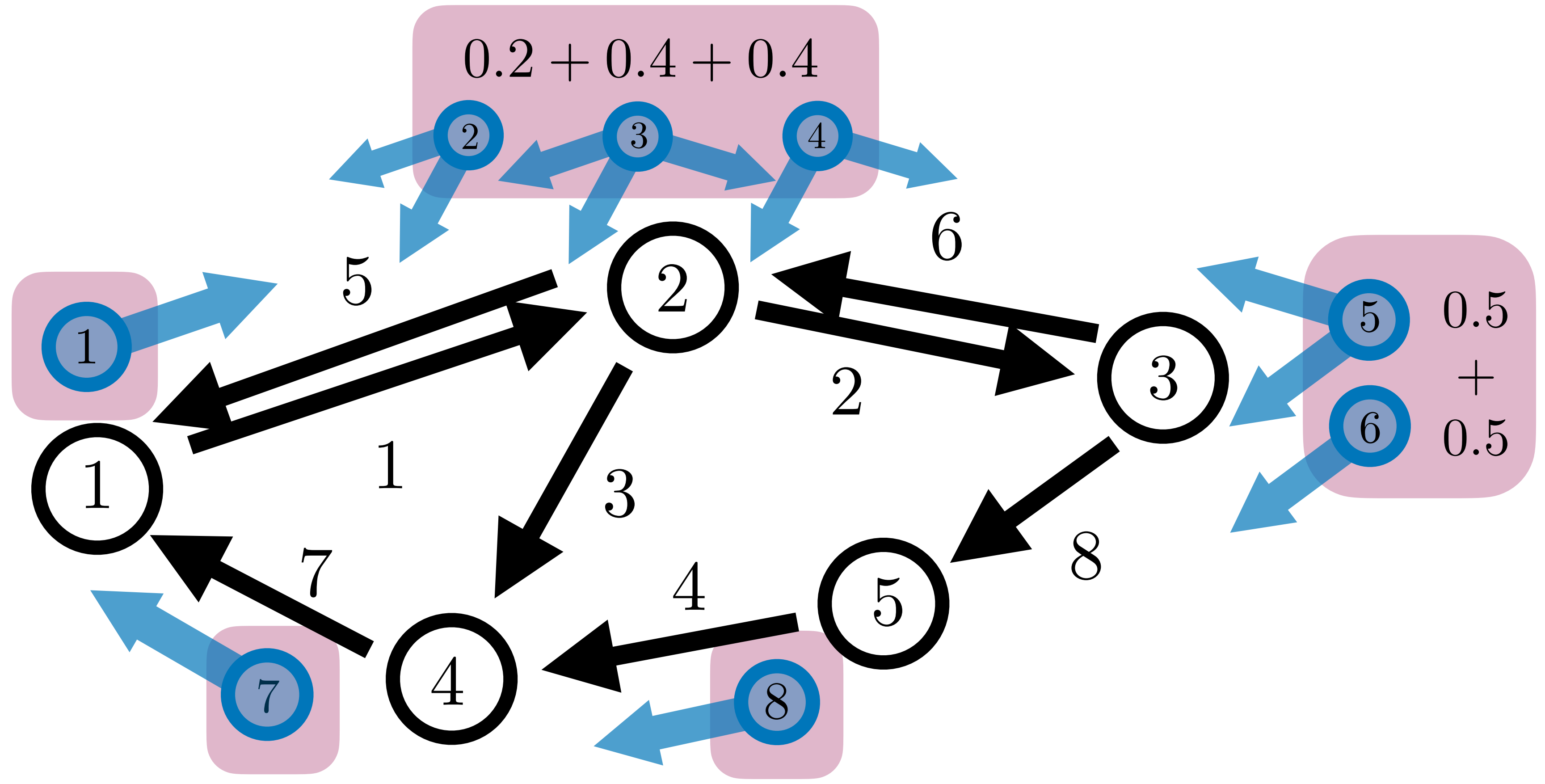

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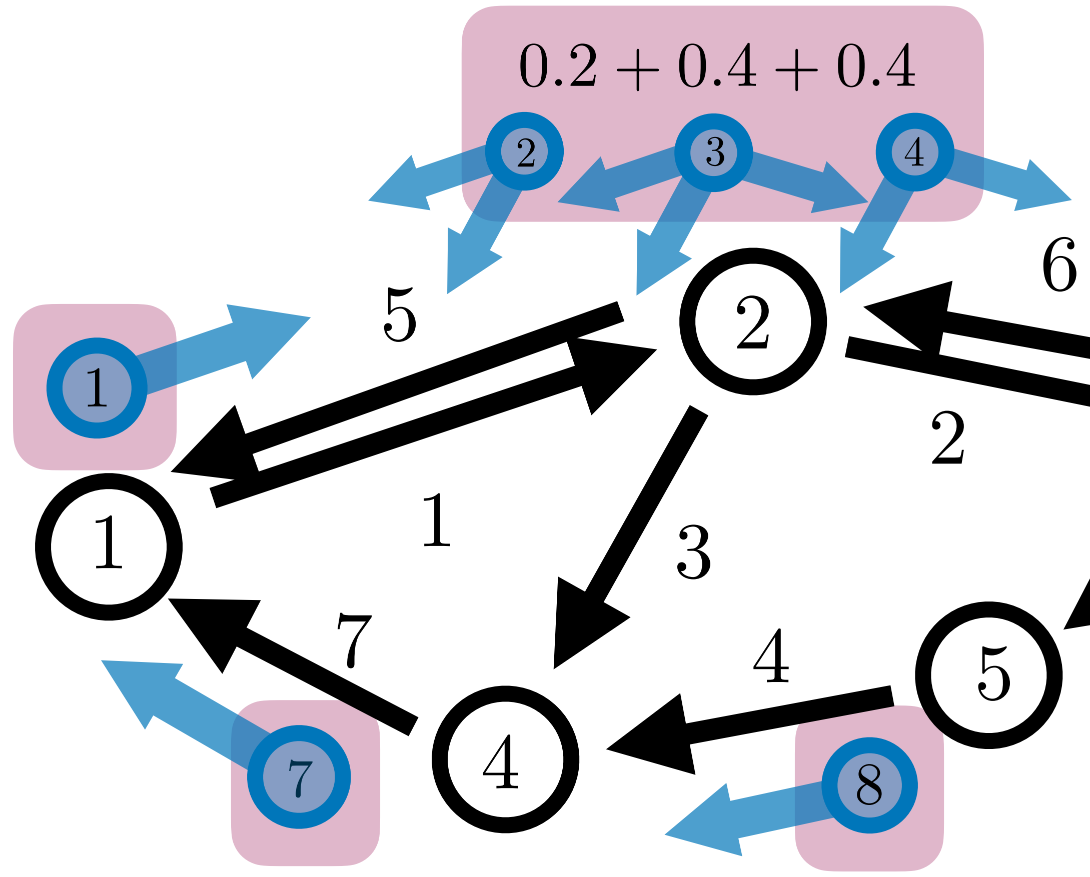

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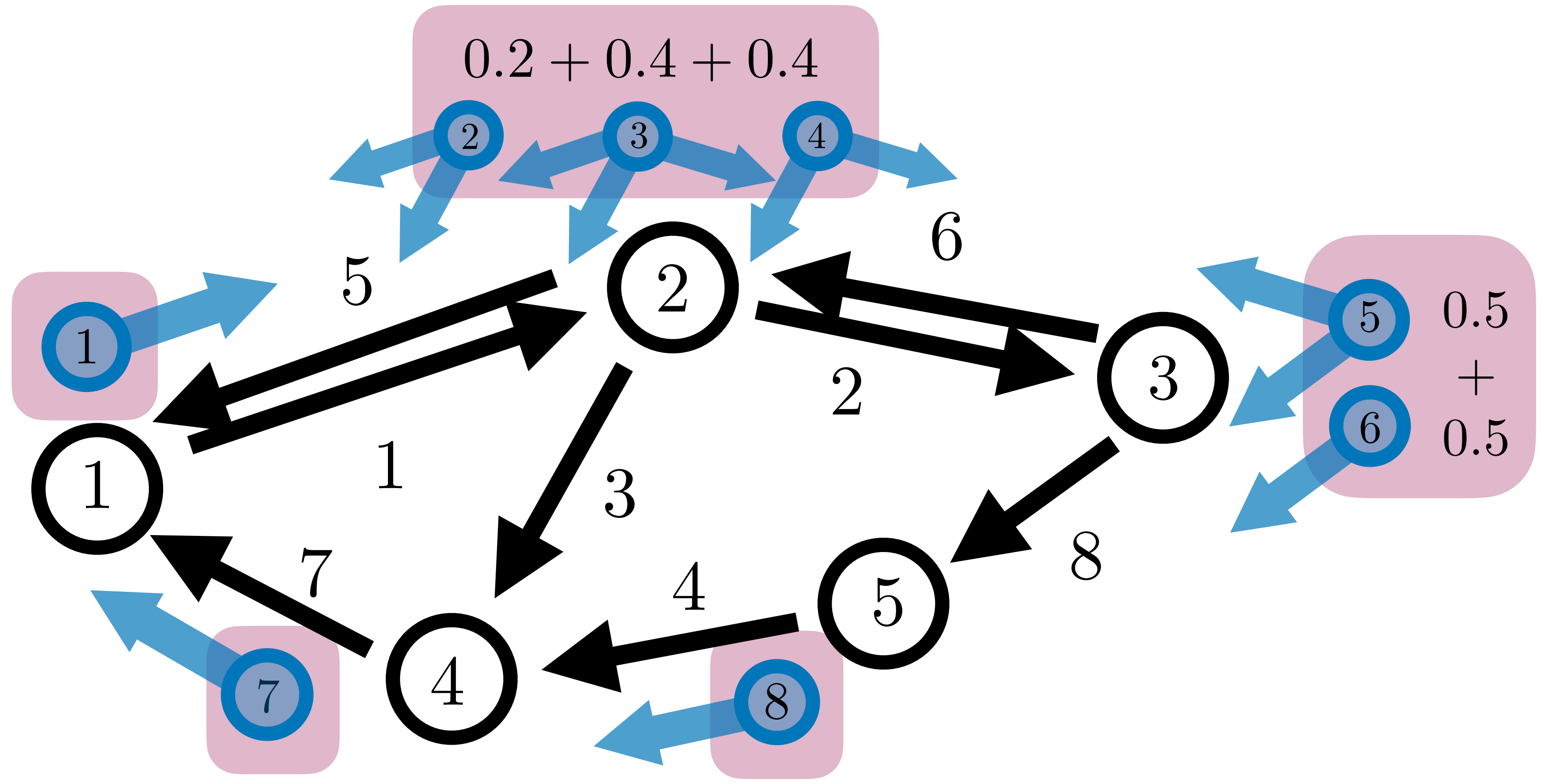

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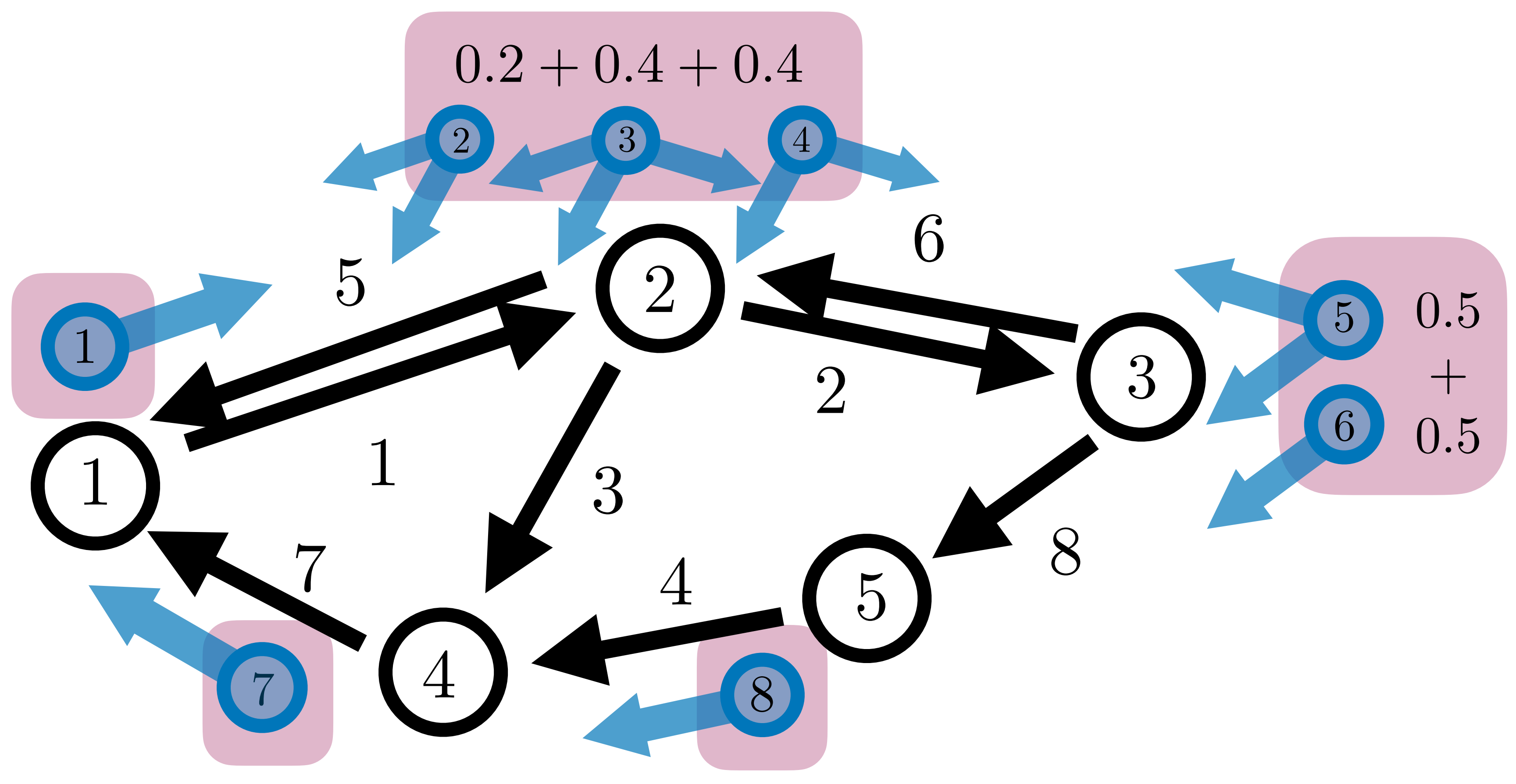

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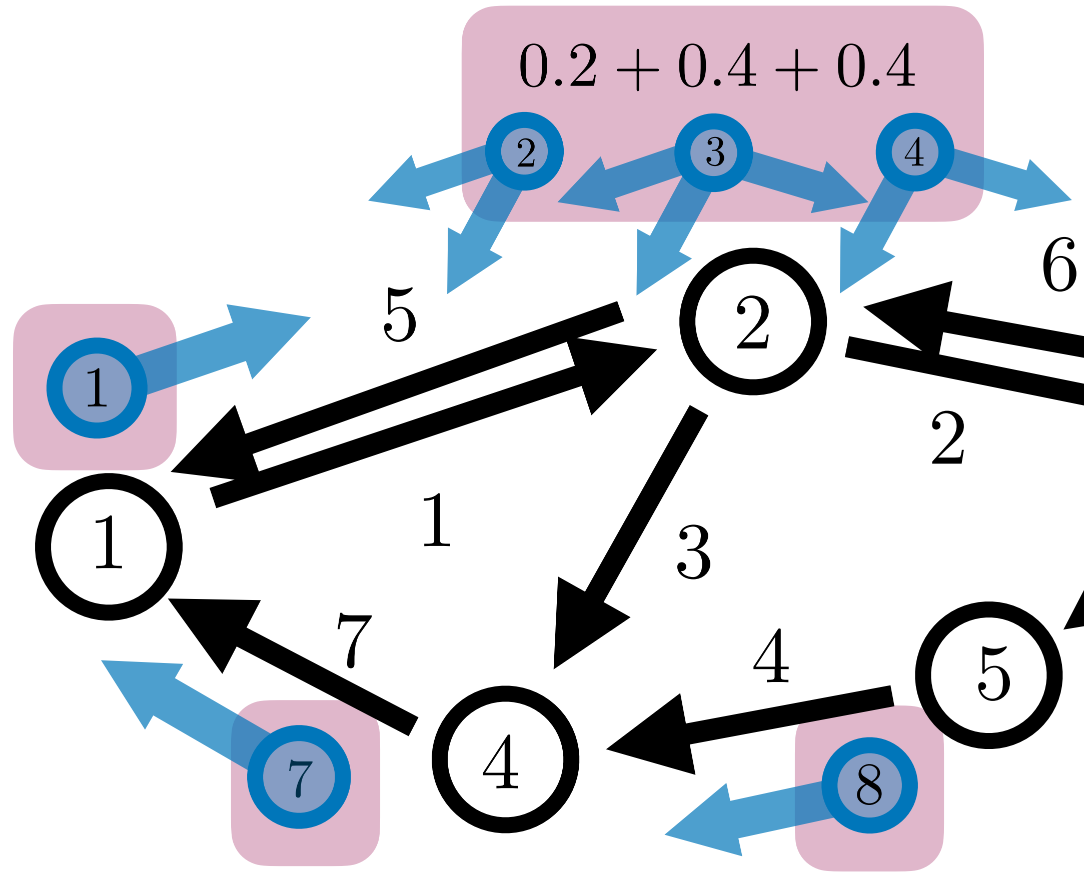

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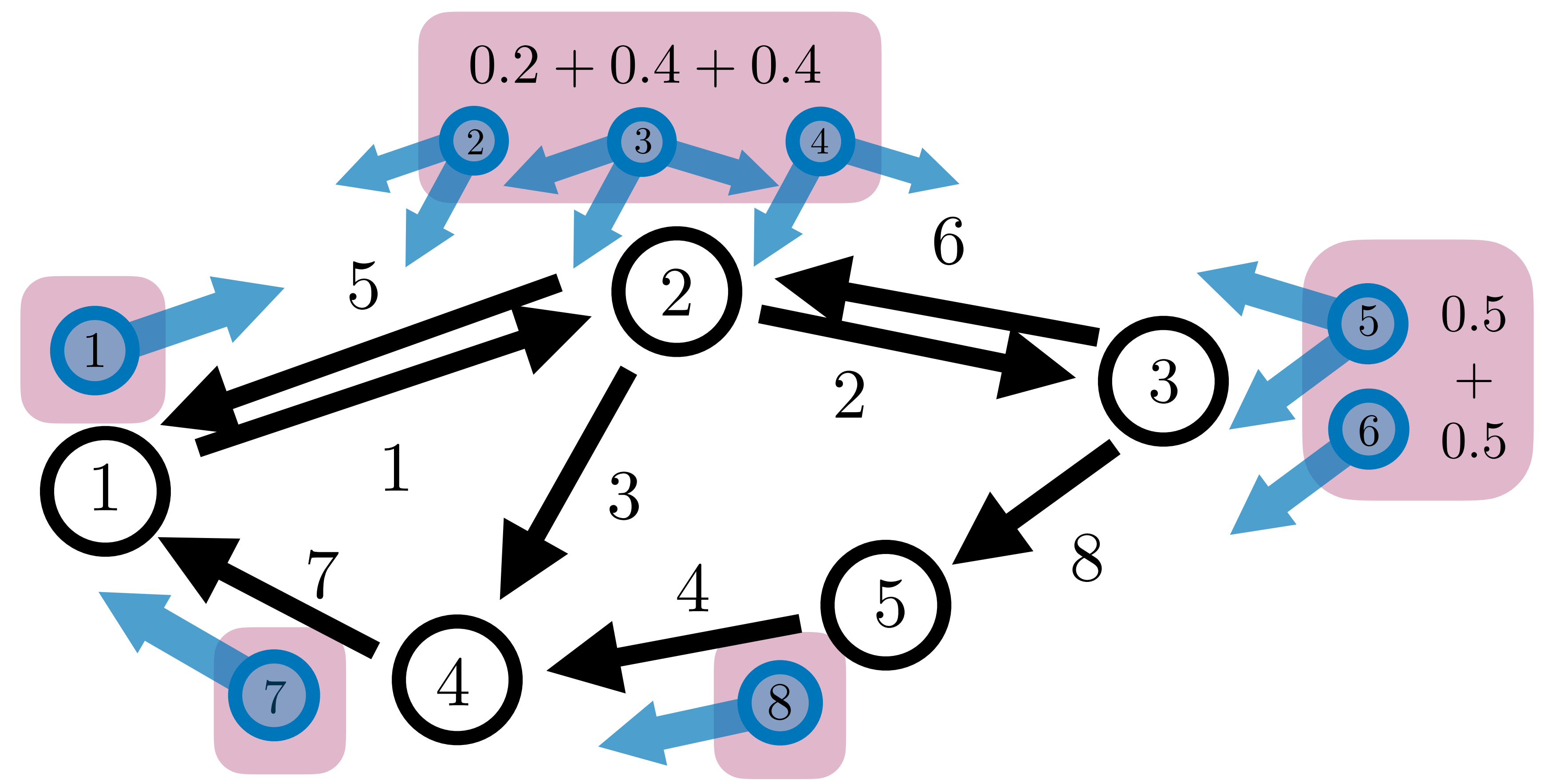

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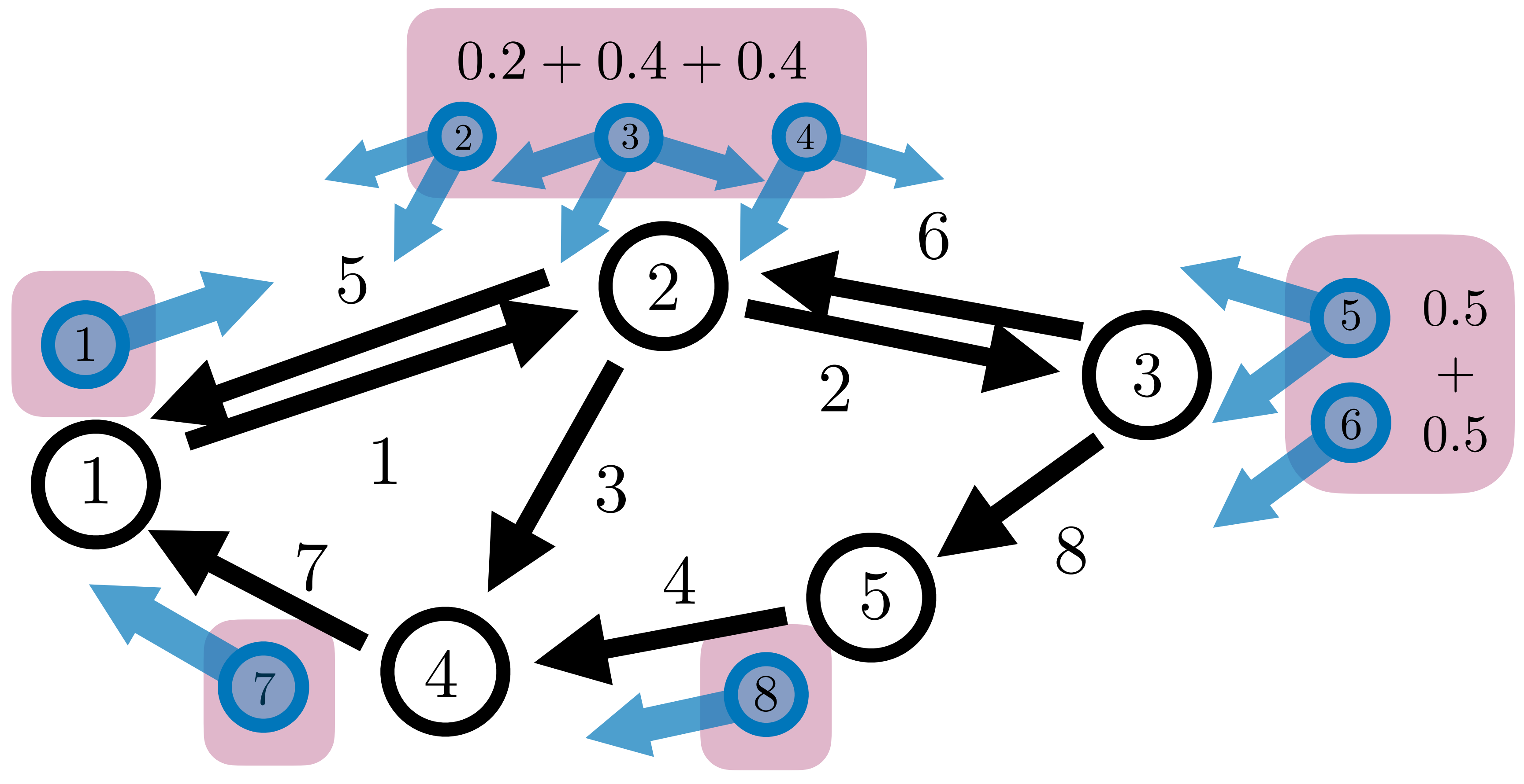

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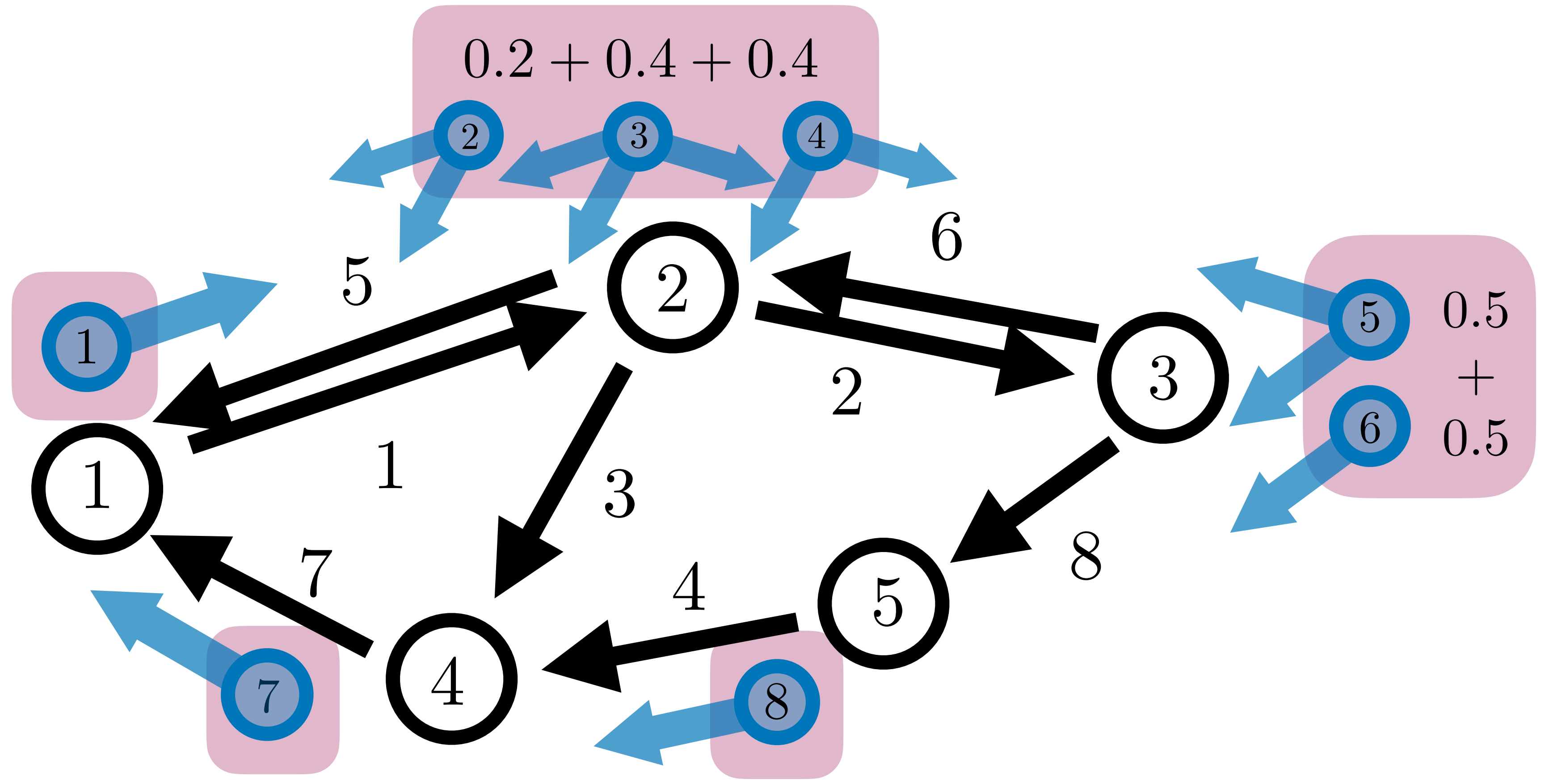

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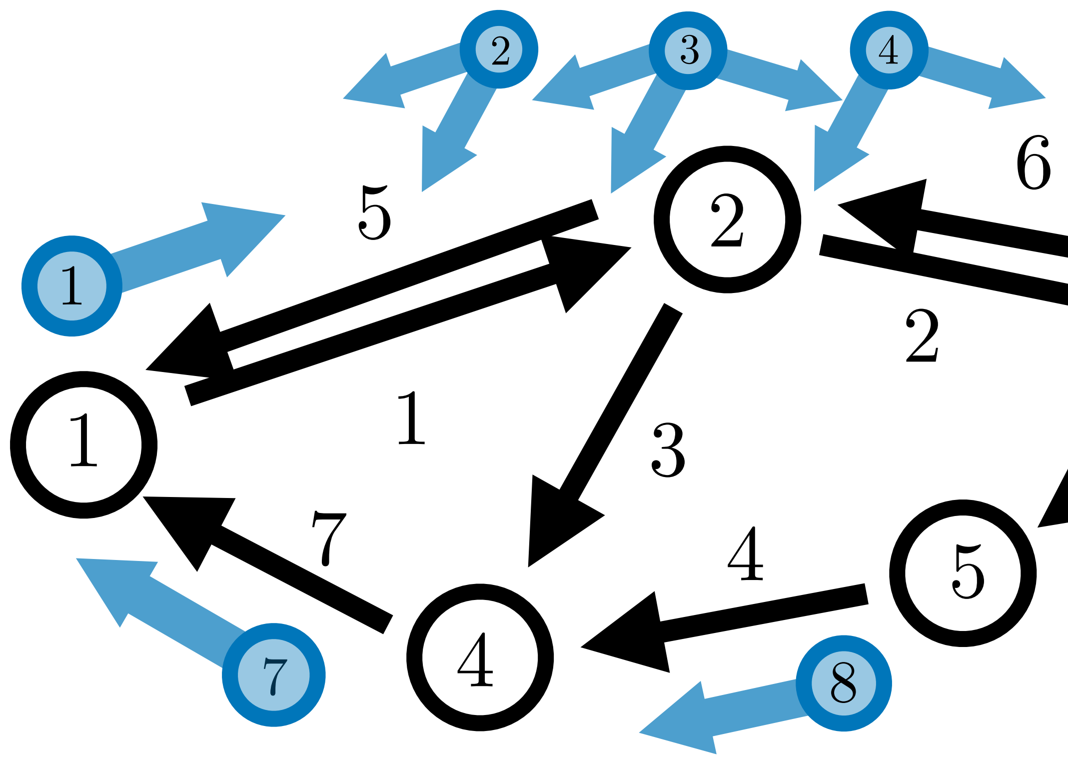

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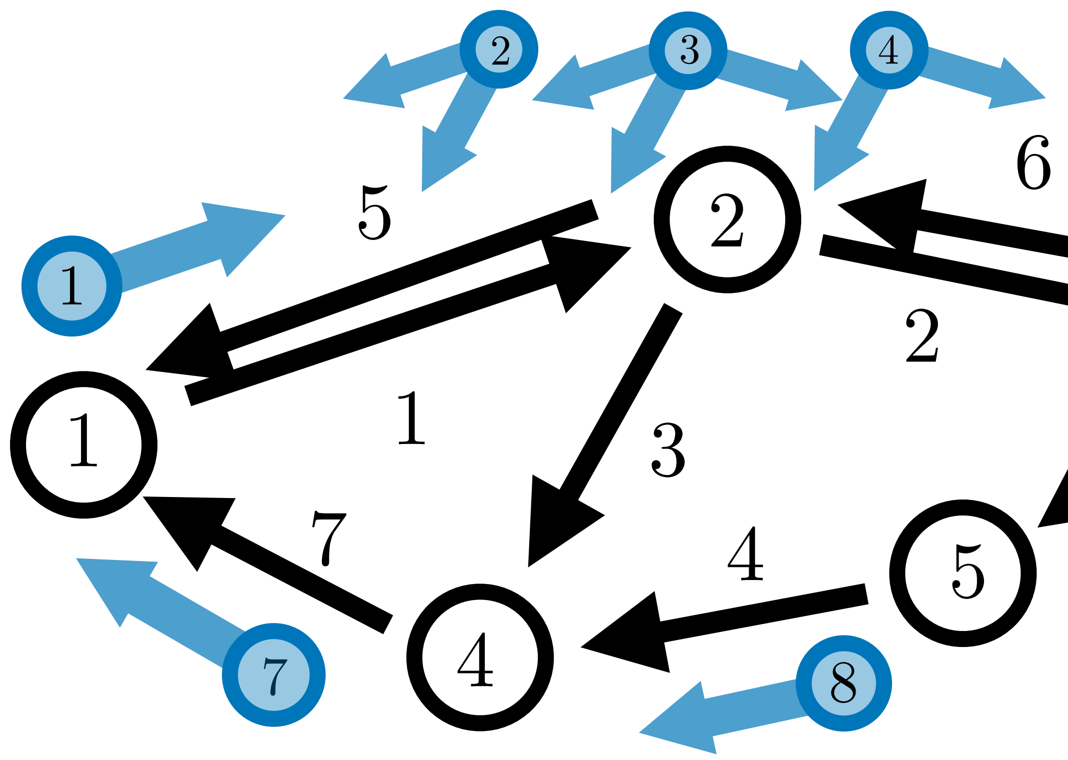

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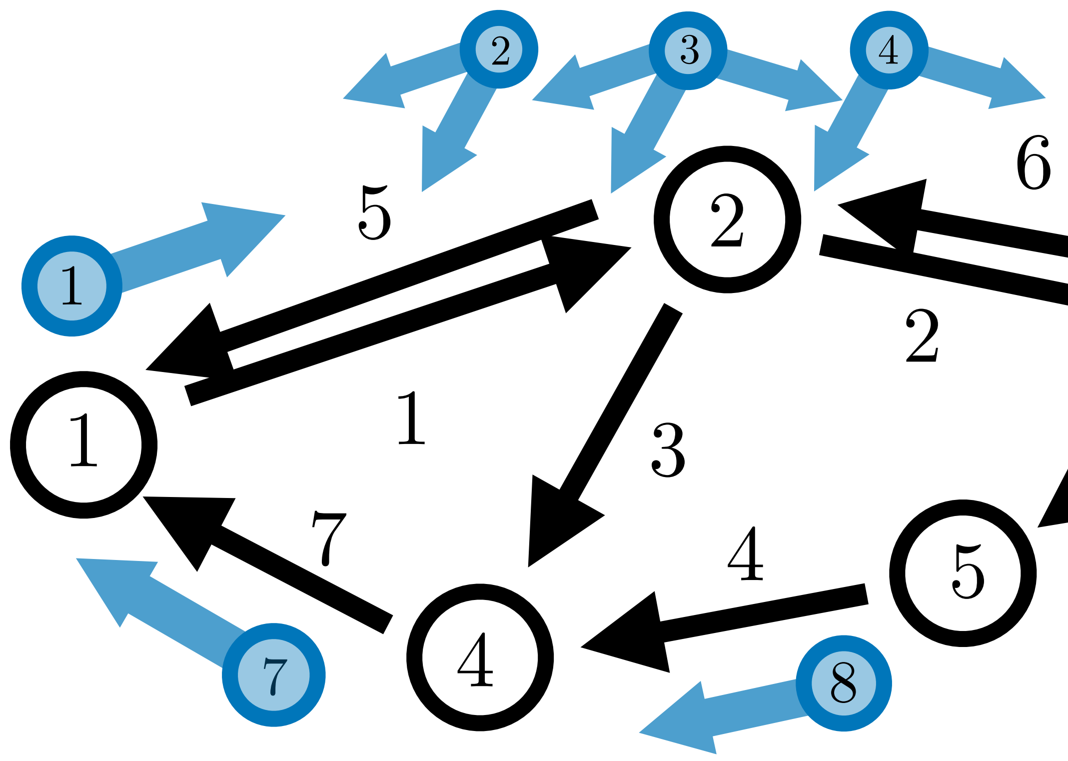

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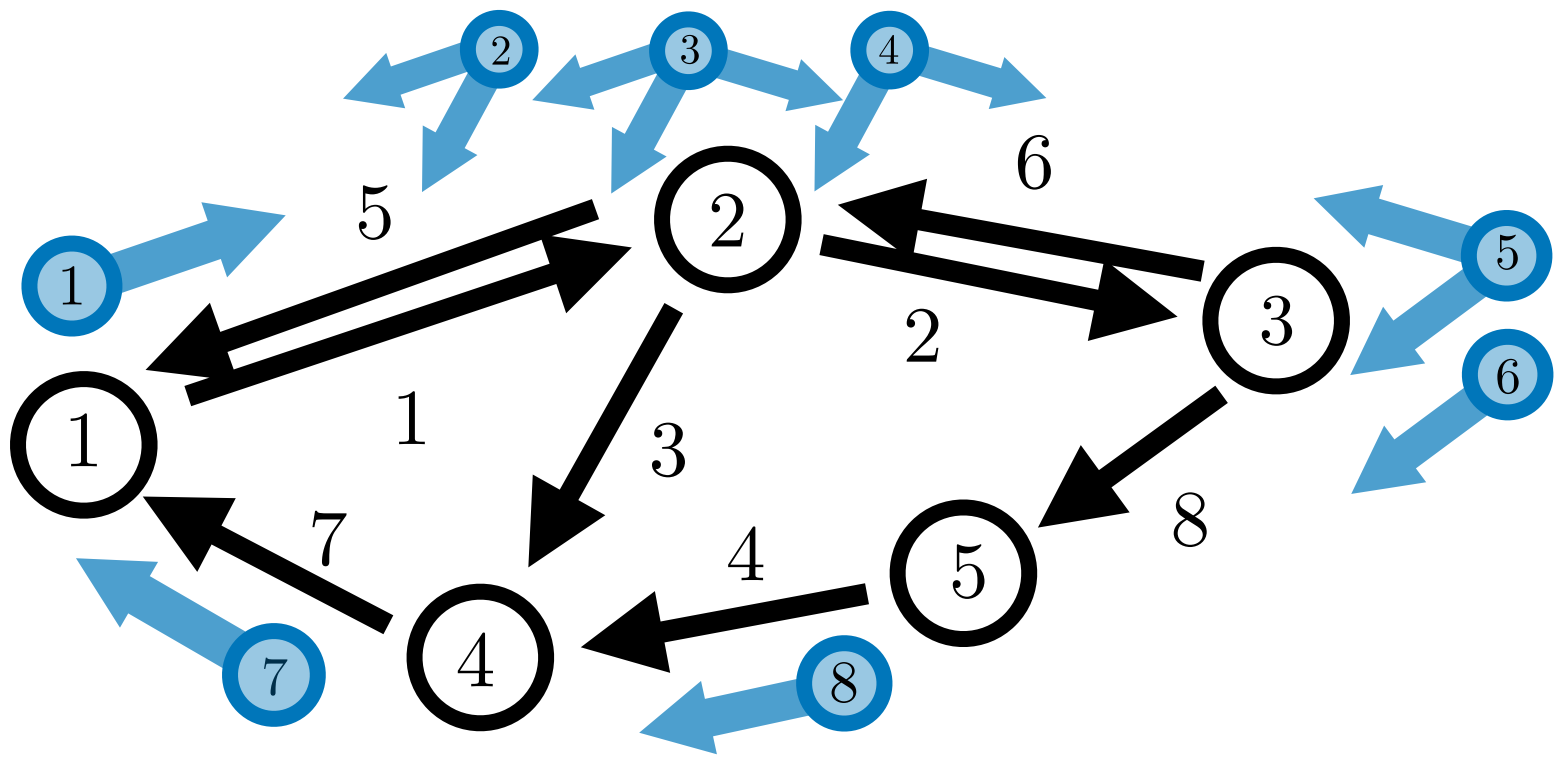

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$\lambda \in \mathbb{R}$ **optimal average reward**

Markov Decision Processes

Graph:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

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Incidence Matrices

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Policy

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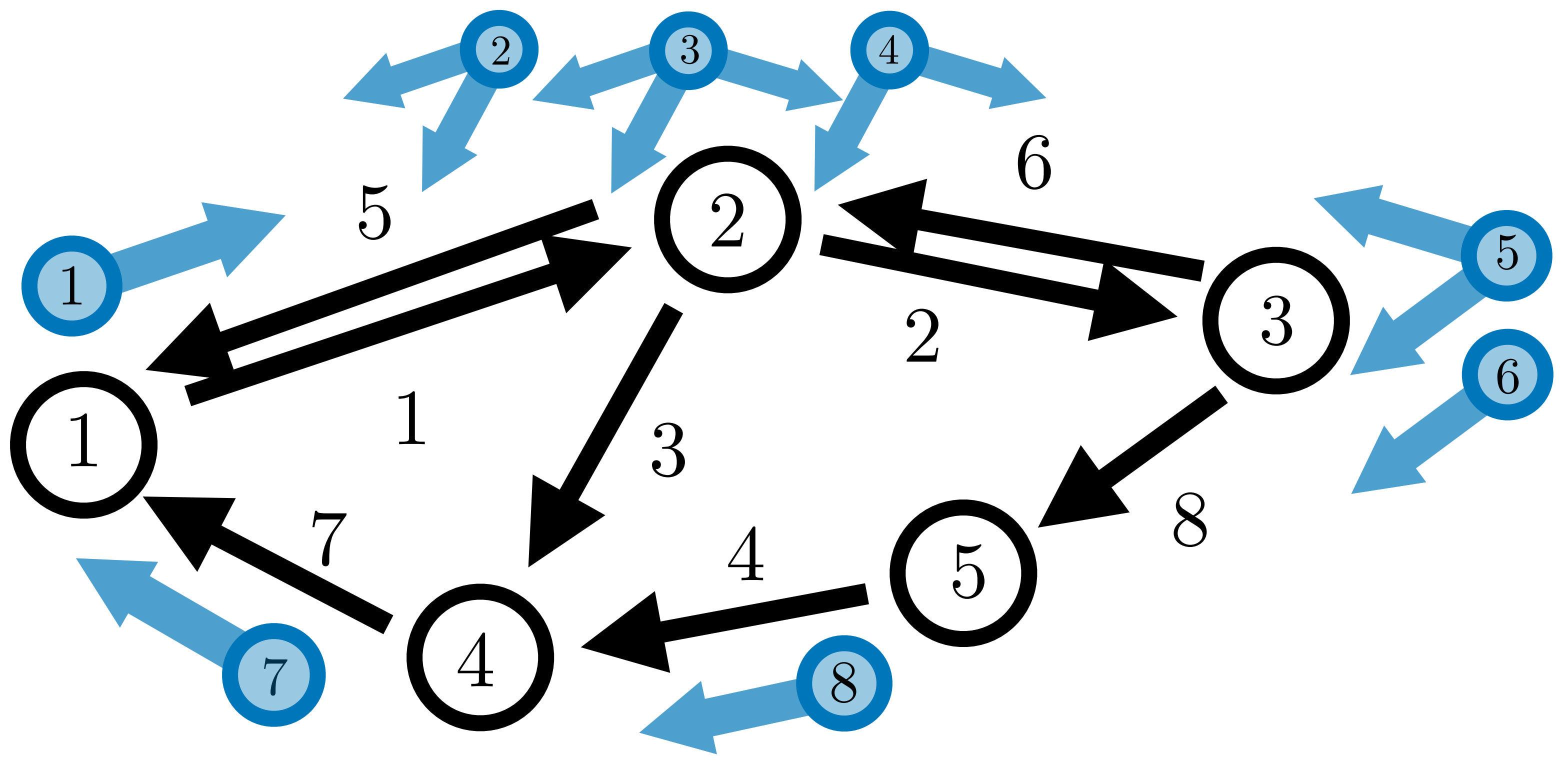

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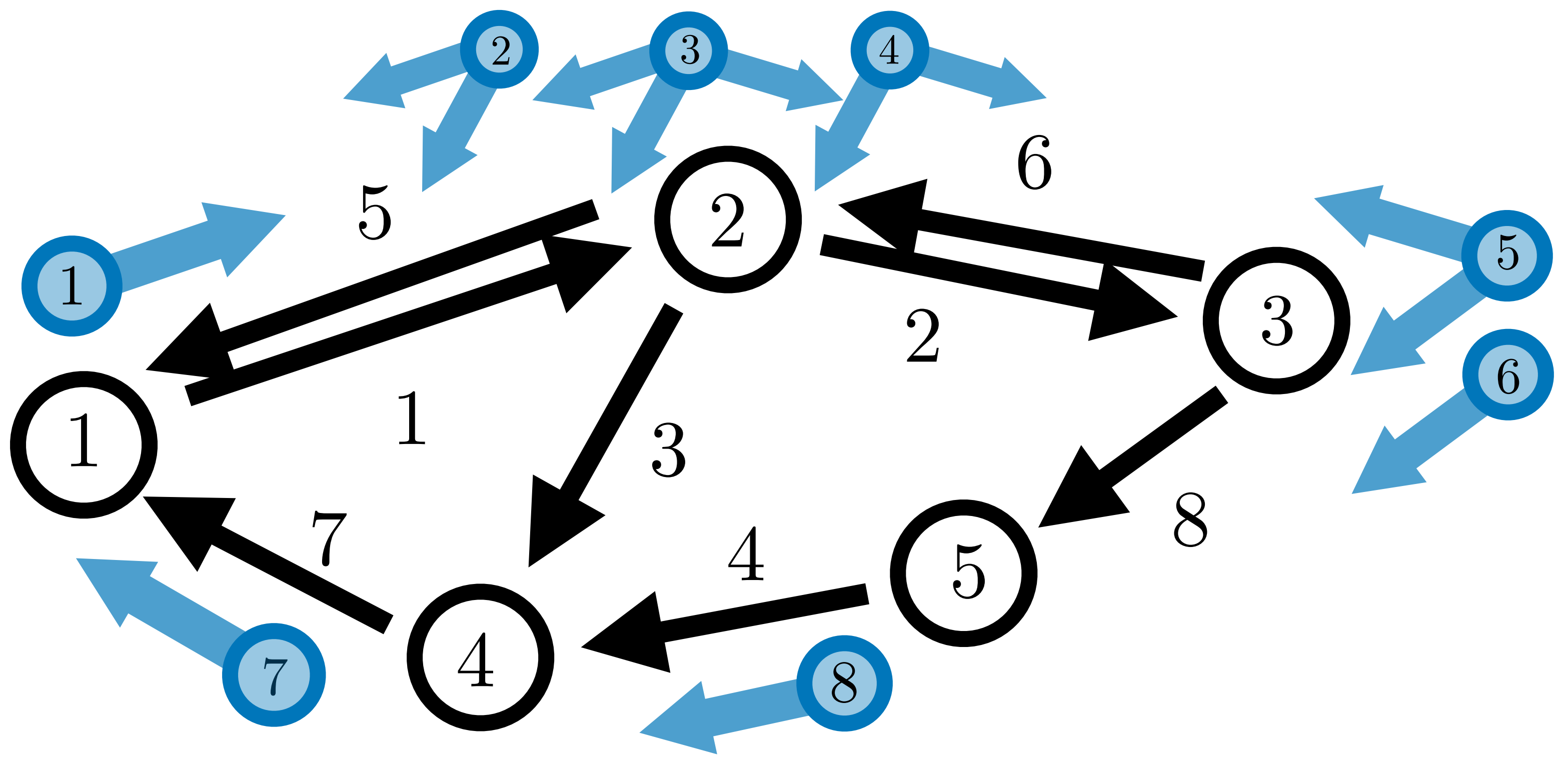

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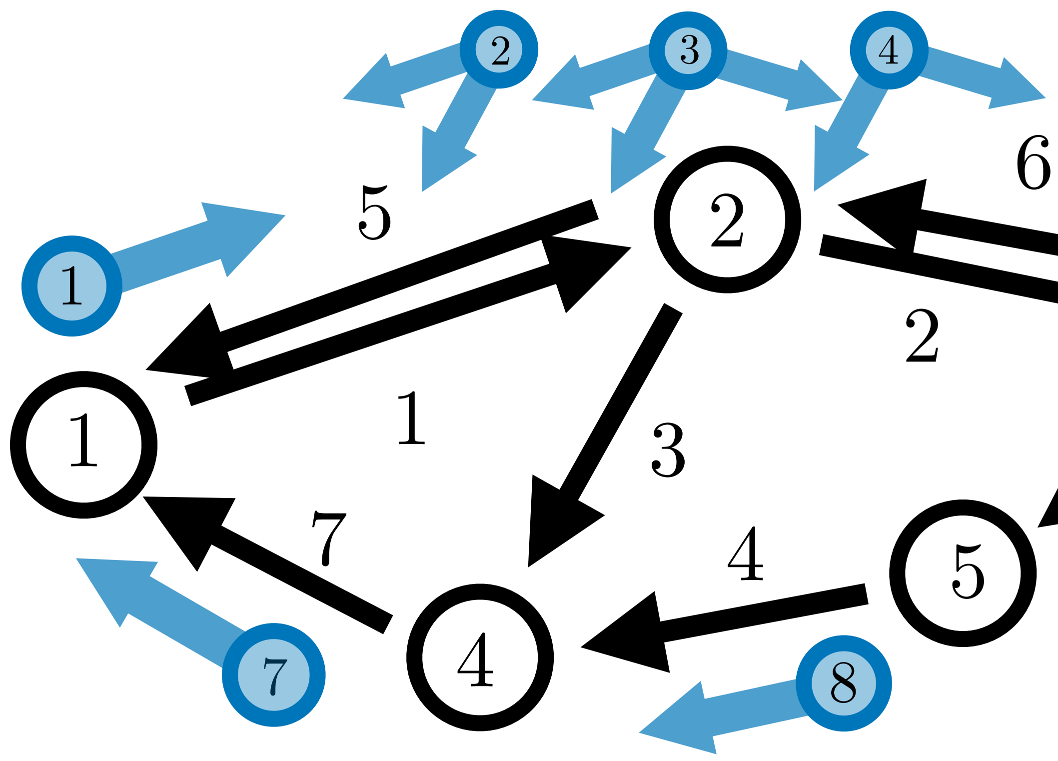

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Complementary slackness $\mu^T x = 0$

“No inefficient action is used”

Markov Decision Processes

Graph:

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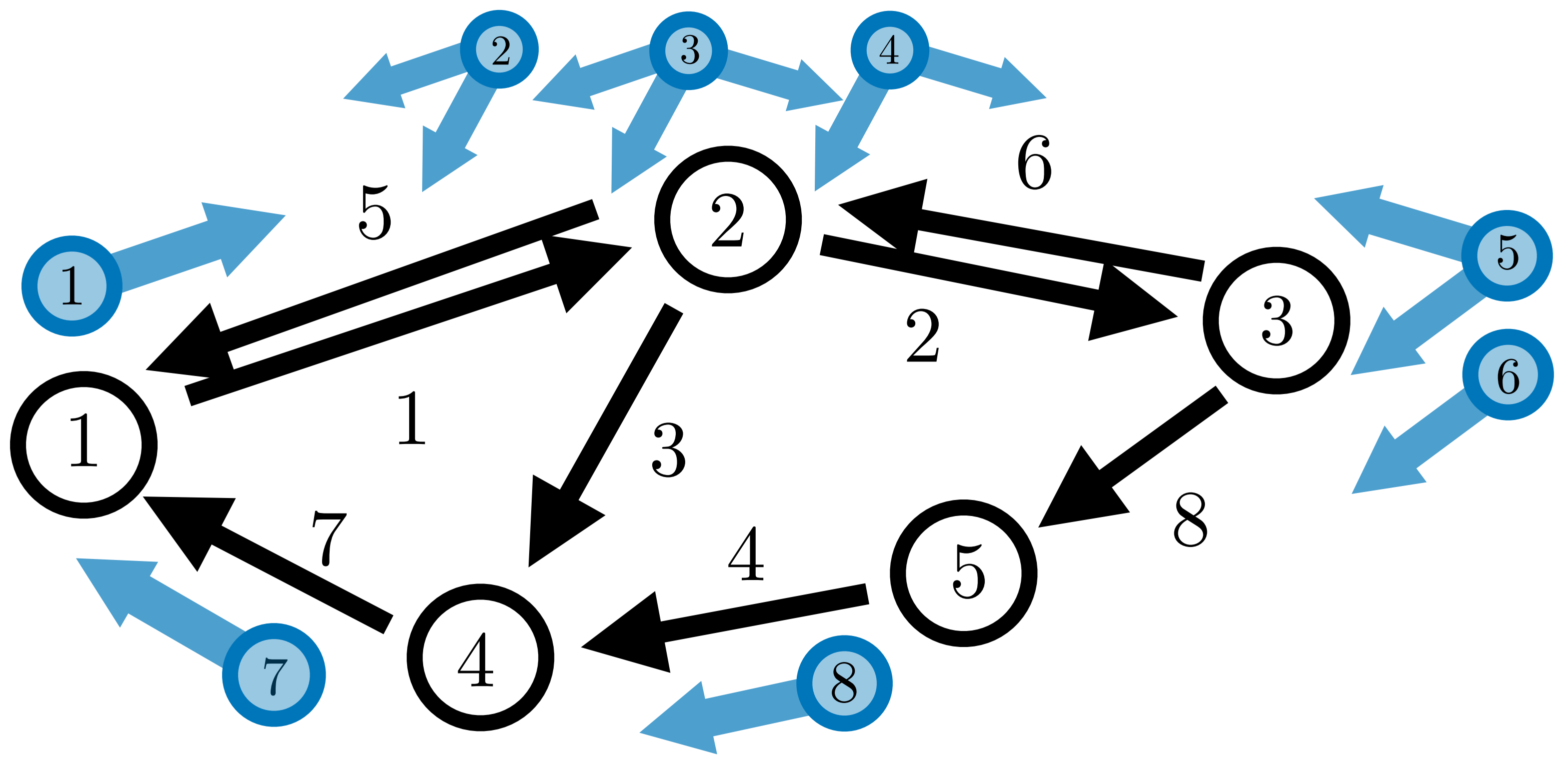

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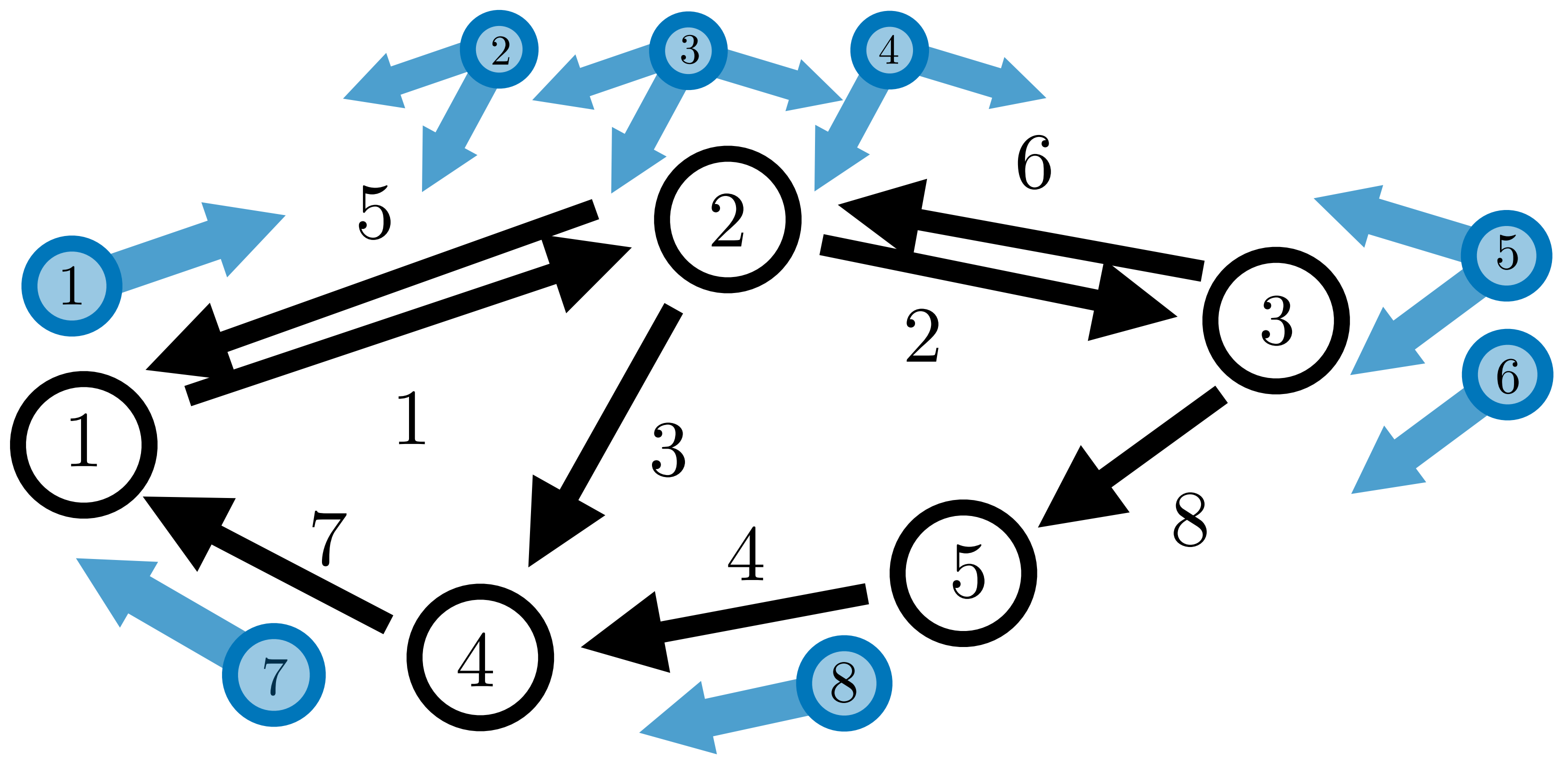

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Q-value $q^T = v^T P$

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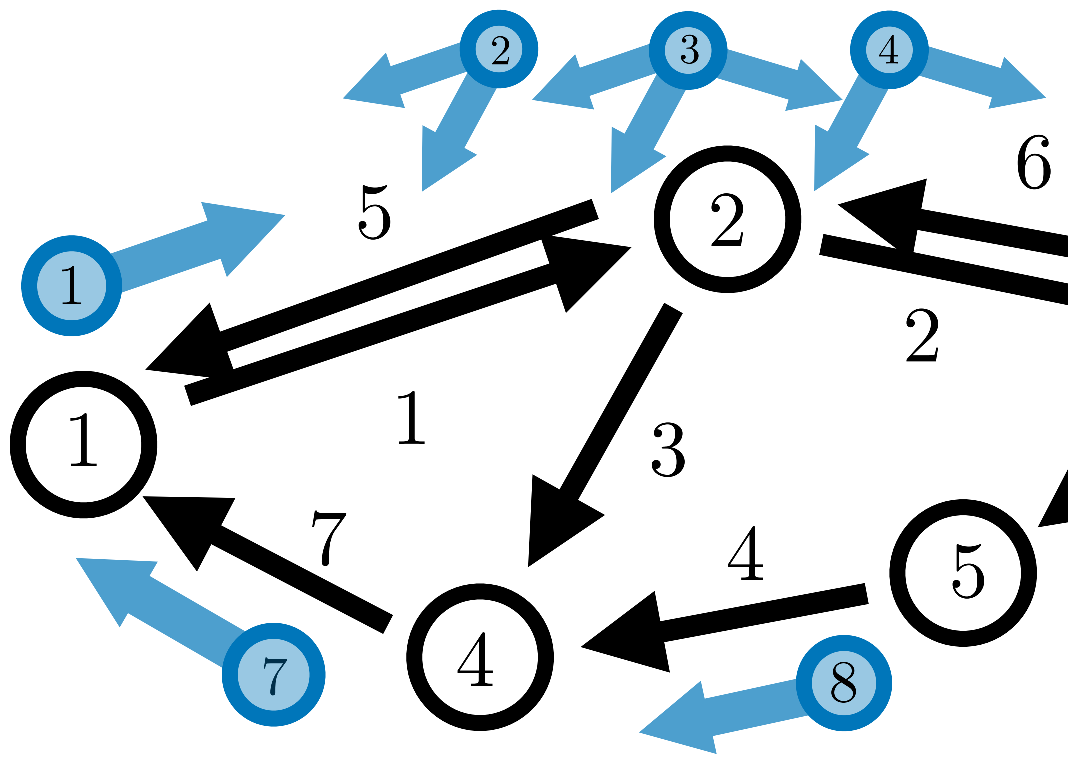

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Optimization of Discounted Rewards

$$\max_x \quad r^T x \ = \sum_{t=0}^{\infty} \gamma^t r^T x_t$$

s.t. $[\ E_{\mathcal{A}} - \gamma P\ ] x = [\ E_{\text{out}} - \gamma E_{\text{in}}\ ] W x = (1-\gamma) z_0$ $v \in \mathbb{R}^{|\mathcal{S}|}$

$\mathbf{1}^T x = 1,$ $\lambda \in \mathbb{R}$ $x \geq 0$ $\mu \in \mathbb{R}_+^{|\mathcal{A}|}$

$\lambda \in \mathbb{R}$	**optimal average reward**
$v \in \mathbb{R}^{\|\mathcal{S}\|}$	**discounted value on states**
$\mu \in \mathbb{R}_+^{\|\mathcal{A}\|}$	**inefficiency of ea. action**

Discounted Bellman Equation $v^T E_{\mathcal{A}} = r^T + \gamma v^T P - \mu^T$

Markov Decision Processes

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Incidence Matrices

$E \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ $E = E_{\text{in}} - E_{\text{out}}$

$$\left[\, E \,\right]_{ve} = \begin{cases} -1 & ; \text{ if } e \text{ out of } v \\ 1 & ; \text{ if } e \text{ into } v \\ 0 & ; \text{ otherwise} \end{cases}$$

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Markov Decision Process

Actions $a \in \mathcal{A}$ total actions $\mathcal{A} = \cup_{s \in \mathcal{S}} \mathcal{A}_s$

$a \in \mathcal{A}_s$ actions from ea. state

For each action: $\text{Prob}(s'|s,a)$ Probability of transitioning to state s' from state s

Transition Kernel

$P \in \mathbb{R}^{|\mathcal{S}| \times |\mathcal{A}|}$

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Policy

$\Pi \in \mathbb{R}^{|\mathcal{A}| \times |\mathcal{S}|}$

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$$\left[\, \Pi \,\right]_{as} = \begin{cases} \text{Prob}(a|s) & ; \text{ prob. of taking } a \text{ given being in } s \\ 0 & \text{otherwise} \end{cases}$$

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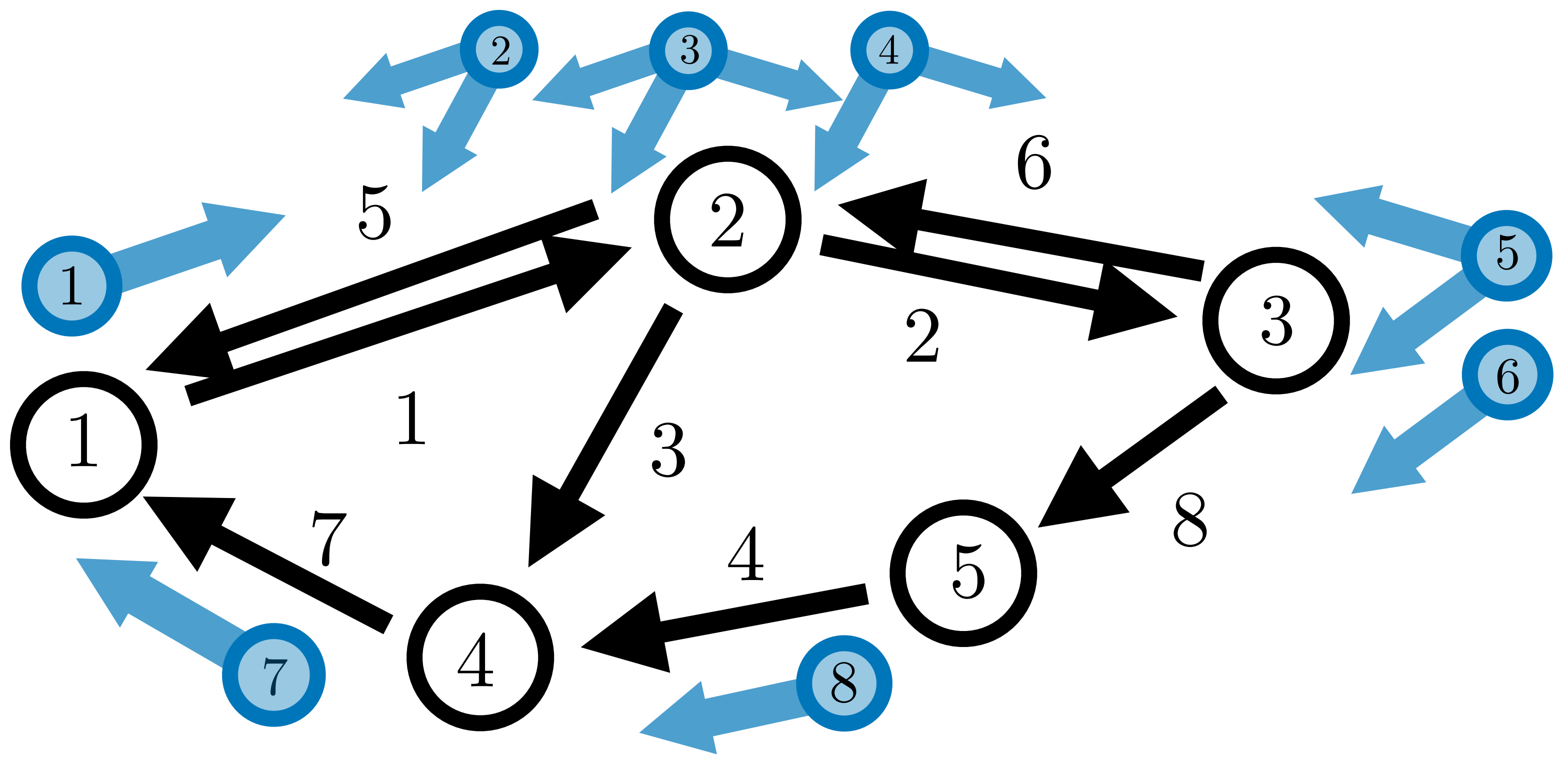

$E_{\mathcal{A}} = E_{\text{out}} W$

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Optimization of Discounted Rewards

$$\max_x \quad r^T x \; = \sum_{t=0}^{\infty} \gamma^t r^T x_t$$

$$\text{s.t.} \quad \left[\, E_{\mathcal{A}} - \gamma P \,\right] x = \left[\, E_{\text{out}} - \gamma E_{\text{in}} \,\right] W x = (1-\gamma) z_0 \qquad v \in \mathbb{R}^{|\mathcal{S}|}$$

~~$\mathbf{1}^T x = 1, \quad \lambda \in \mathbb{R}$~~ $x \geq 0 \quad \mu \in \mathbb{R}_+^{|\mathcal{A}|}$

~~$\lambda \in \mathbb{R}$ optimal average reward~~

$v \in \mathbb{R}^{|\mathcal{S}|}$ **discounted value on states**

$\mu \in \mathbb{R}_+^{|\mathcal{A}|}$ **inefficiency of ea. action**

Discounted Bellman Equation $v^T E_{\mathcal{A}} = r^T + \gamma v^T P - \mu^T$

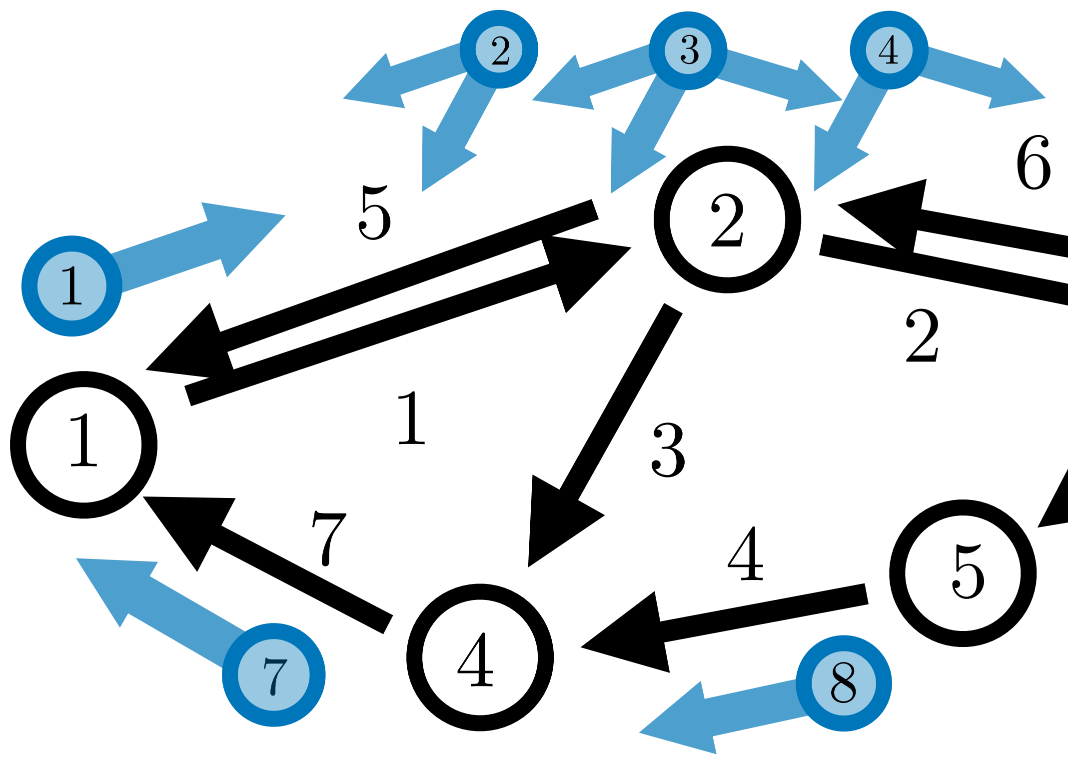
Markov Decision Processes

Graph:

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Vertices $v \in \mathcal{V}$ **States** $s \in \mathcal{S}$ $\mathcal{V} = \mathcal{S}$

Edges $e \in \mathcal{E}$ $e = (v, v')$



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$\mu \in \mathbb{R}_+^{|\mathcal{A}|}$ inefficiency of ea. action

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Discounted Q-value $q^T = v^T P$

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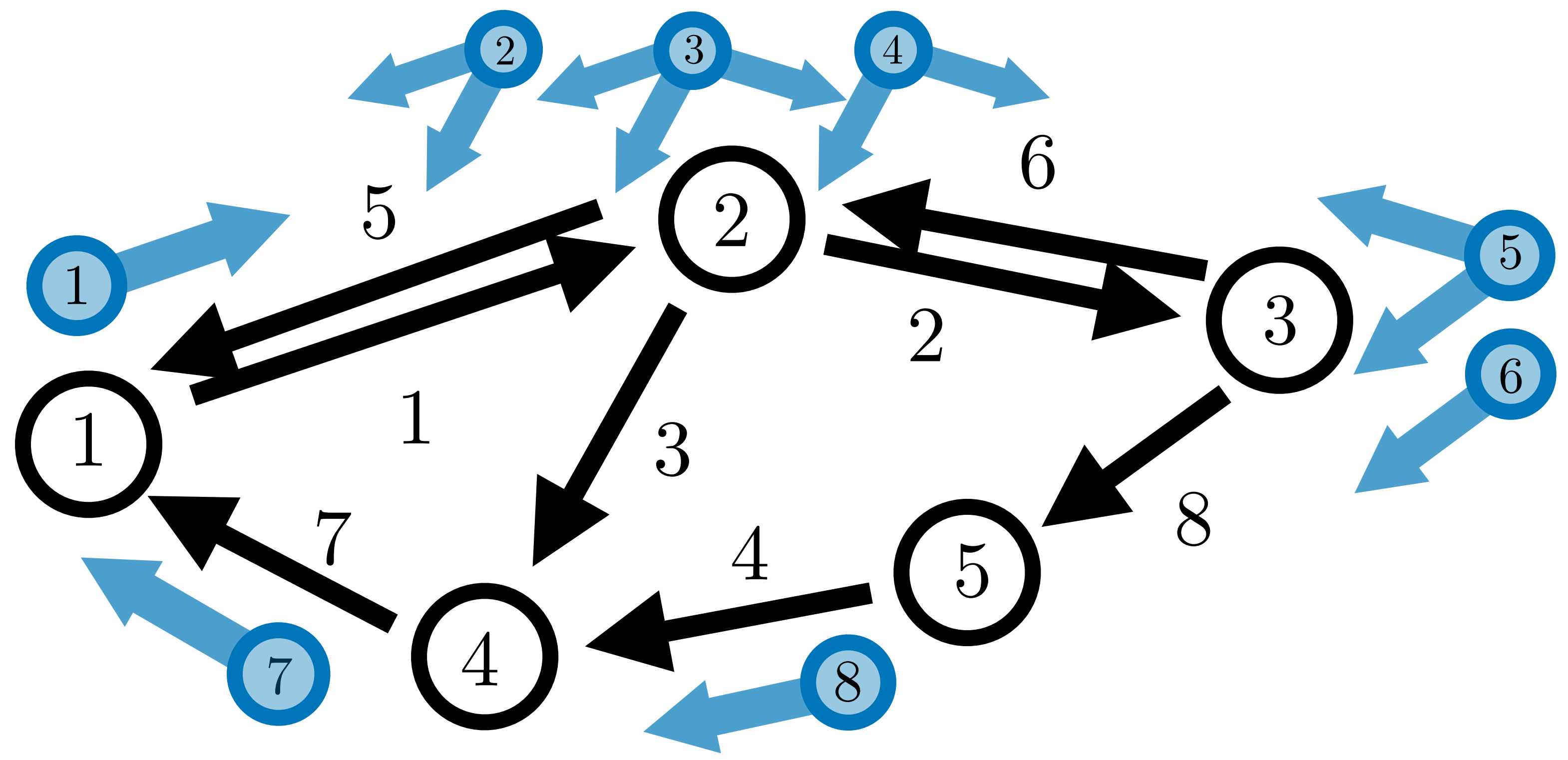

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Optimization of Average Rewards

$$\max_x \quad r^T x$$

$$\text{s.t.} \quad [\ E_{\mathcal{A}} - P\ ]x = [\ E_{\text{out}} - E_{\text{in}}\ ]Wx = 0 \quad v \in \mathbb{R}^{|\mathcal{S}|}$$

$$\mathbf{1}^T x = 1, \quad \lambda \in \mathbb{R} \qquad x \geq 0 \quad \mu \in \mathbb{R}_+^{|\mathcal{A}|}$$

Recovering Policy

$$\Pi_{as} = \frac{x_{sa}}{\sum_{a' \in \mathcal{A}_s} x_{sa'}} \qquad \Pi = \text{dg}(x) E_{\mathcal{A}}^T \text{dg}\big(E_{\mathcal{A}} x\big)^{-1}$$