Dynamics Examples

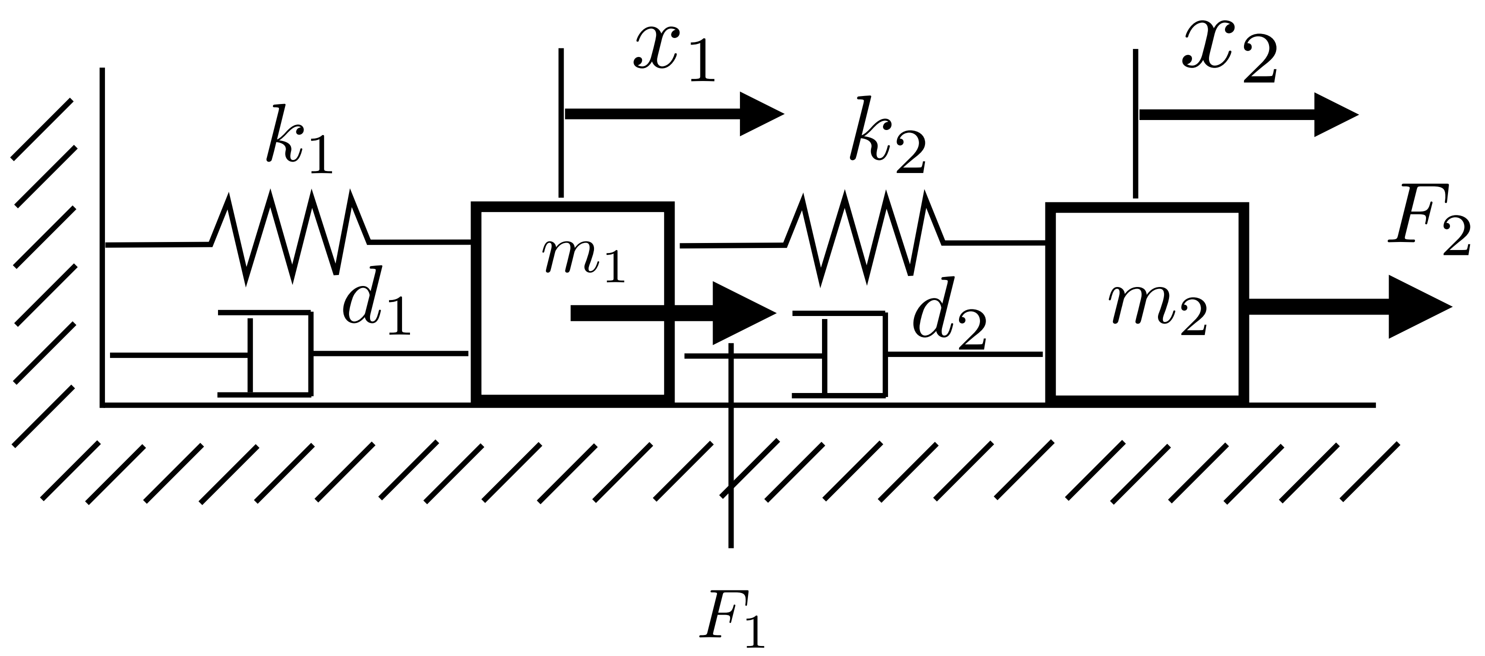
Dynamics & Modeling

Major sources:

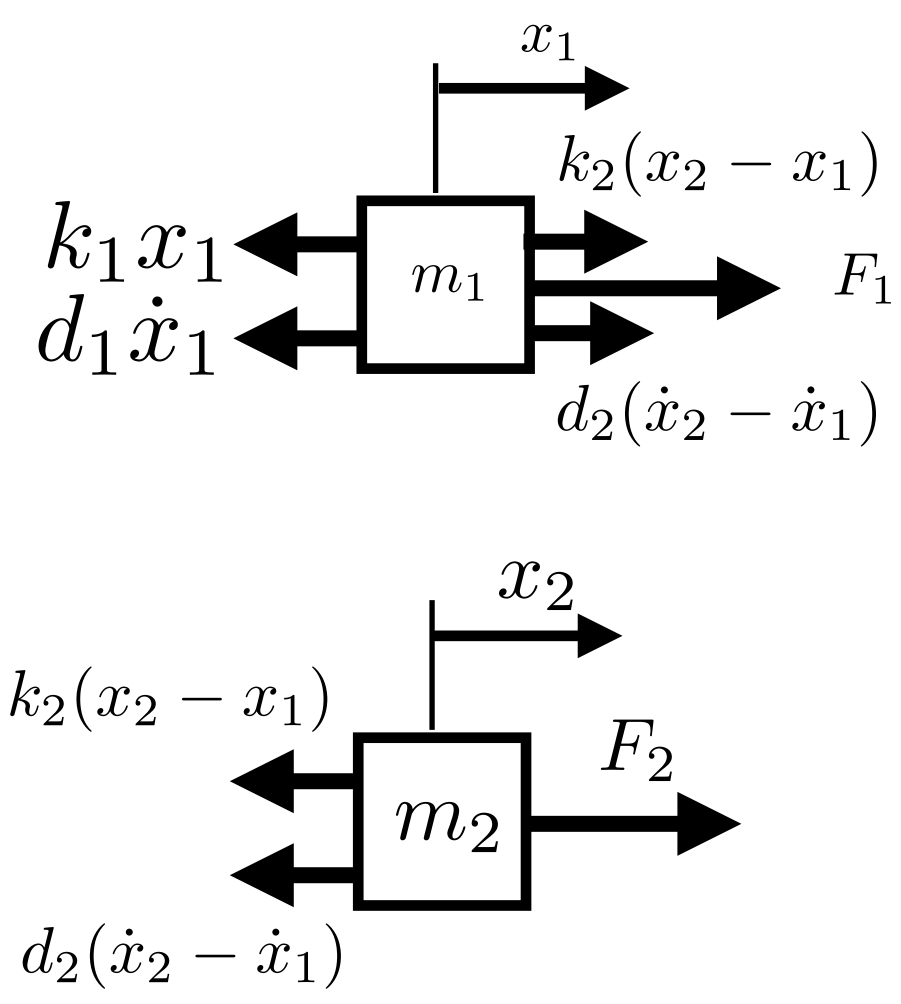
Major sources:

Winter 2022 - Dan Calderone

Example: Two Blocks



Free Body Diagrams



Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Dynamics: $\sum (\text{forces})_x = m\ddot{x}$

Block 1: 1 $F_1 - k_1 x_1 + k_2(x_2 - x_1) - d_1\dot{x}_1 \boxed{+} d_2(\dot{x}_2 - \dot{x}_1) = m_1 \ddot{x}_1$

Block 2: 2 $F_2 - k_2(x_2 - x_1) \boxed{-} d_2(\dot{x}_2 - \dot{x}_1) = m_2 \ddot{x}_2$

State Space Model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \frac{\boxed{-} d_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Linearization

$$\dot{x} = f(x, u)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \frac{\boxed{-} d_2}{m_2} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

Example: Two Blocks

Linearization Stability: Case 1

$$k_1 = k_2 = 1 \qquad m_1 = m_2 = 1$$

$$d_1 = d_2 = 0.001$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 1. \\ -2. & 1. & -0.002 & 0.001 \\ 1. & -1. & 0.001 & -0.001 \end{bmatrix}$$

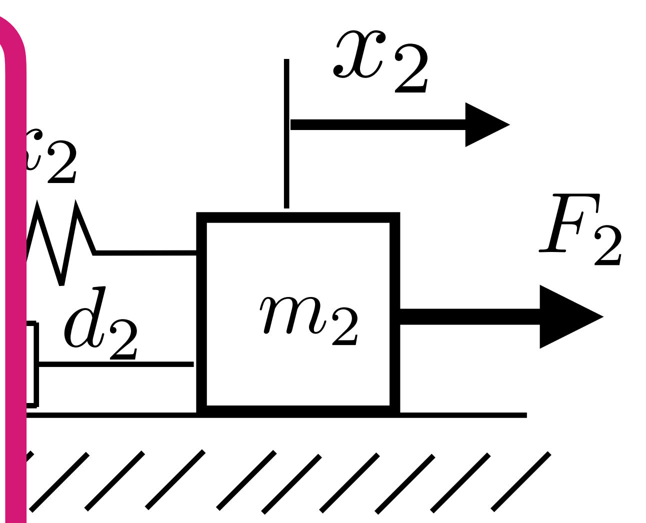
Eigenvalues

$$\lambda_{1,2} = -0.0013 \pm 1.618i$$

$$\lambda_{3,4} = -0.00019 \pm 0.618i$$

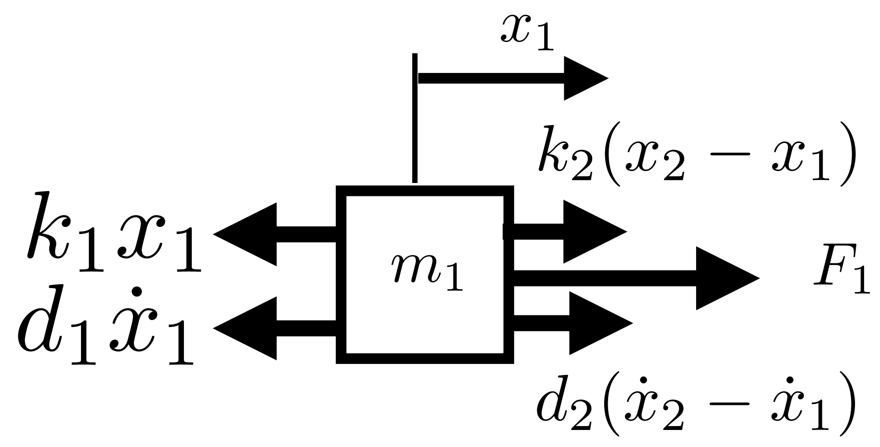
Comments

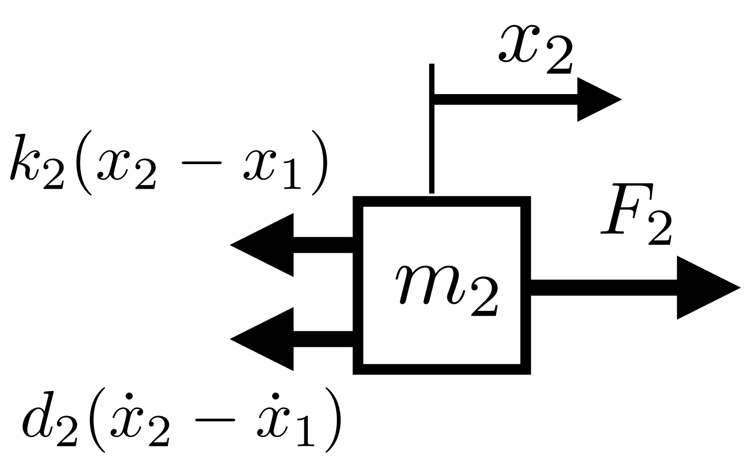
Since the damping is small, the system decays slowly,
ie. The real parts of the eigenvalues are negative with small magnitude
The complex eigenvalues means that the system will oscillate back and forth
(to be expected with springs with not much damping)



$\ldots_1 \ddot{x}_1$

Free Body Diagrams





…zation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \boxed{-}\frac{d_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\dot{x} = f(x, u)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \boxed{-}\frac{d_2}{m_2} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

Example: Two Blocks

Linearization Stability: Case 1

$$k_1 = k_2 = 1 \qquad m_1 = m_2 = 1$$

$$d_1 = d_2 = 100$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 1. \\ -2. & 1. & -200 & 100 \\ 1. & -1. & 100 & -100 \end{bmatrix}$$

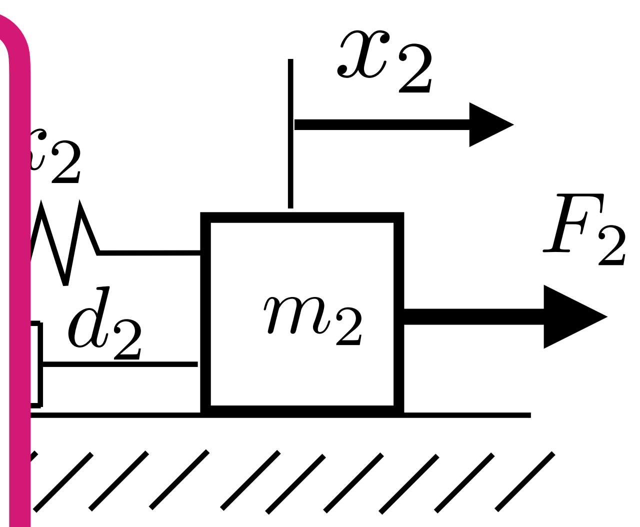
Eigenvalues

$\lambda_1 = -261.8 \qquad \lambda_2 = -38.19$

$\lambda_{3,4} = -0.0100$

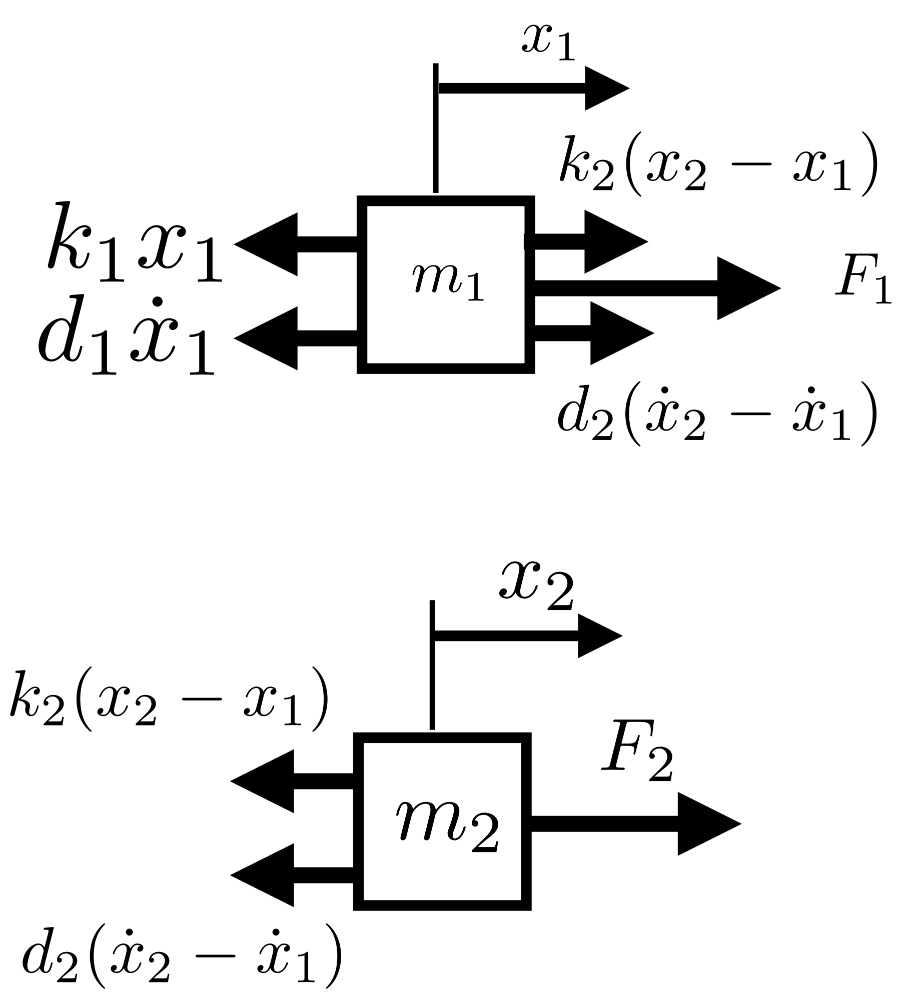
Comments

Since the damping is large, the system is over damped and the system does not oscillate (all real eigenvalues). There are some slow modes (3 & 4) but the system settles down very quickly due to the large damping, ie. Two of the eigenvalues are very negative.



$_1\ddot{x}_1$

Free Body Diagrams



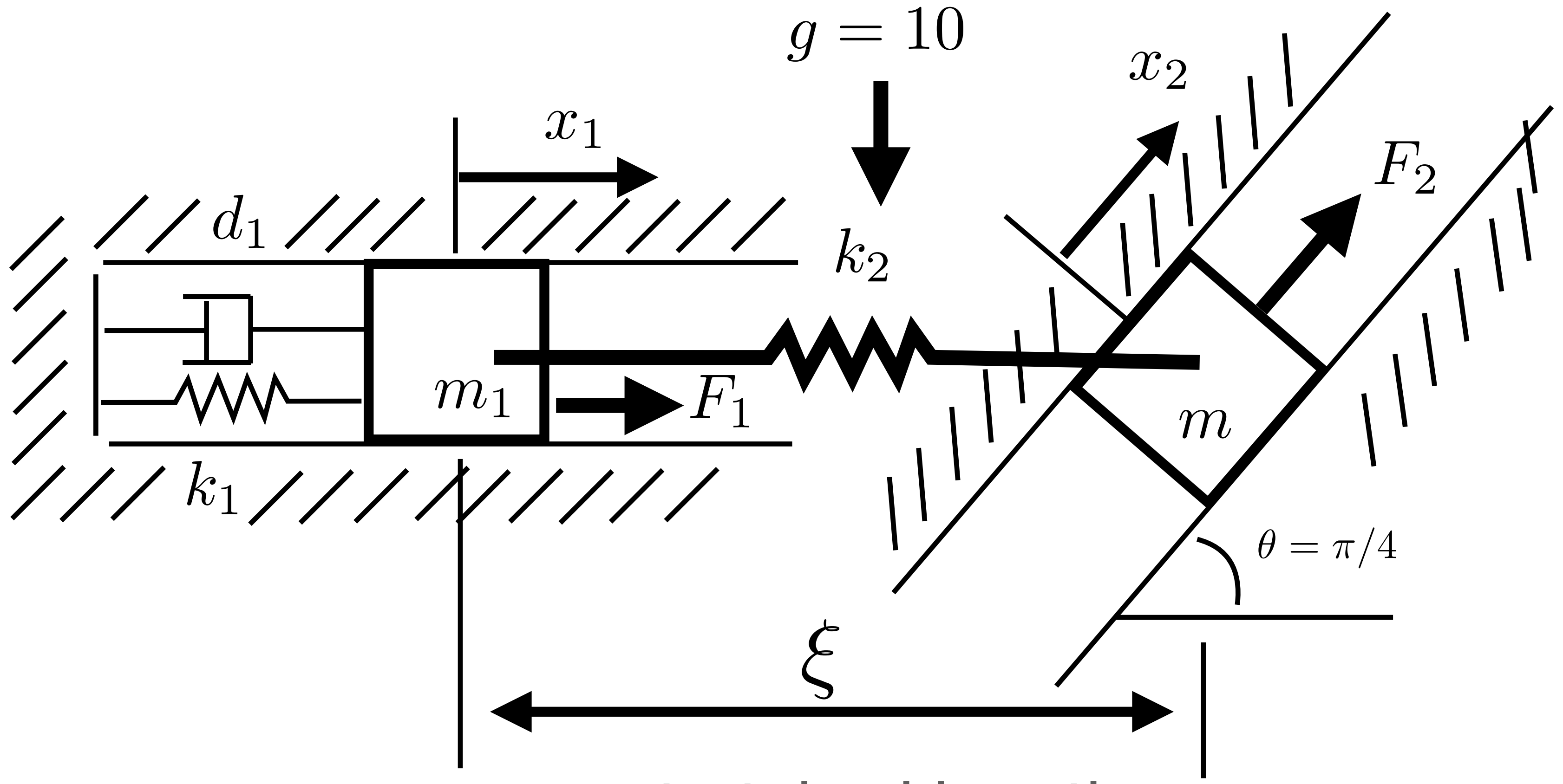
zation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \boxed{-}\frac{d_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

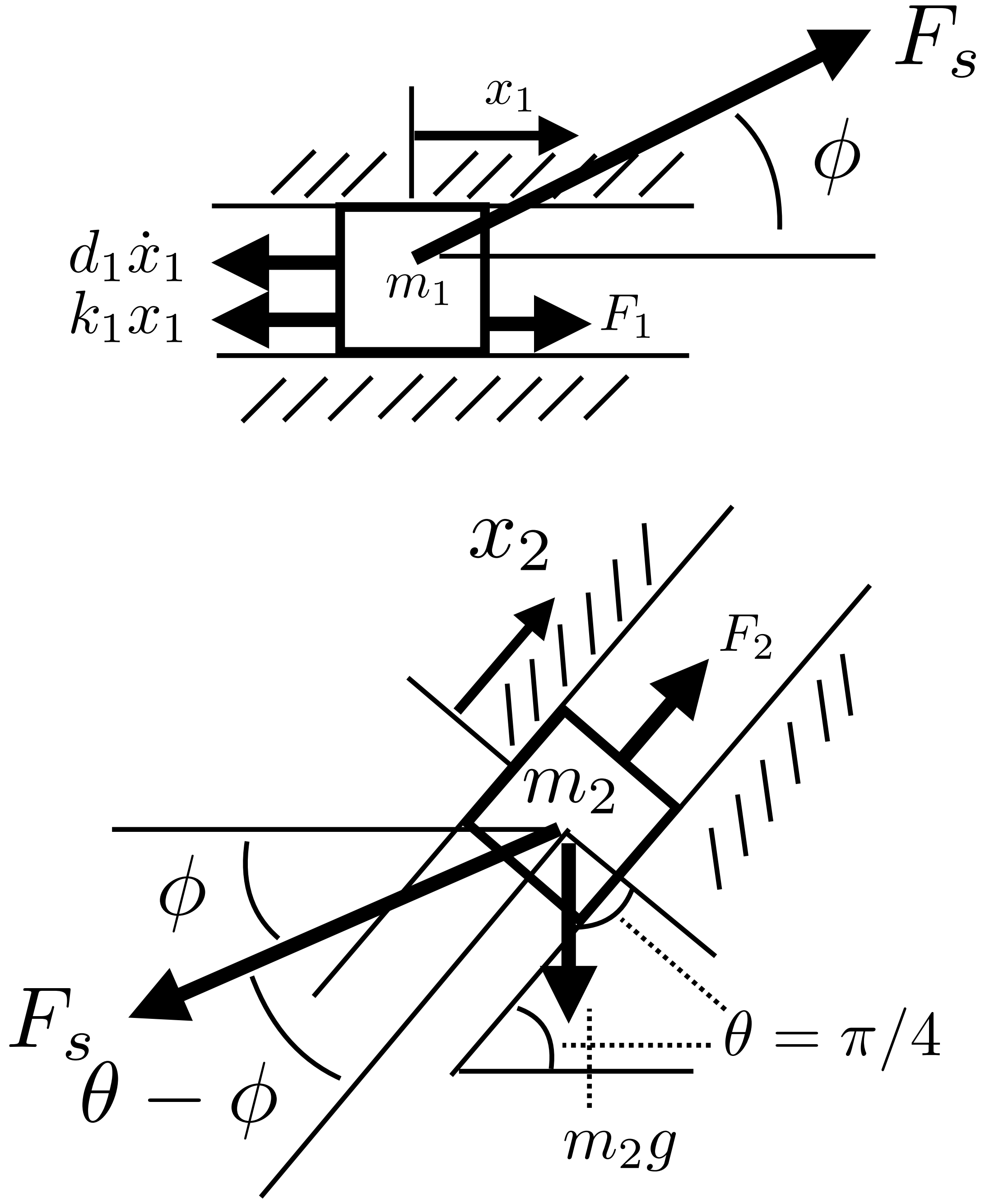
$$\dot{x} = f(x, u)$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d_1 \boxed{-} d_2}{m_1} & \frac{\boxed{+} d_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \boxed{\frac{+d_2}{m_2}} & \boxed{-}\frac{d_2}{m_2} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

Example: Two Blocks - Oblique Connection



Free Body Diagrams

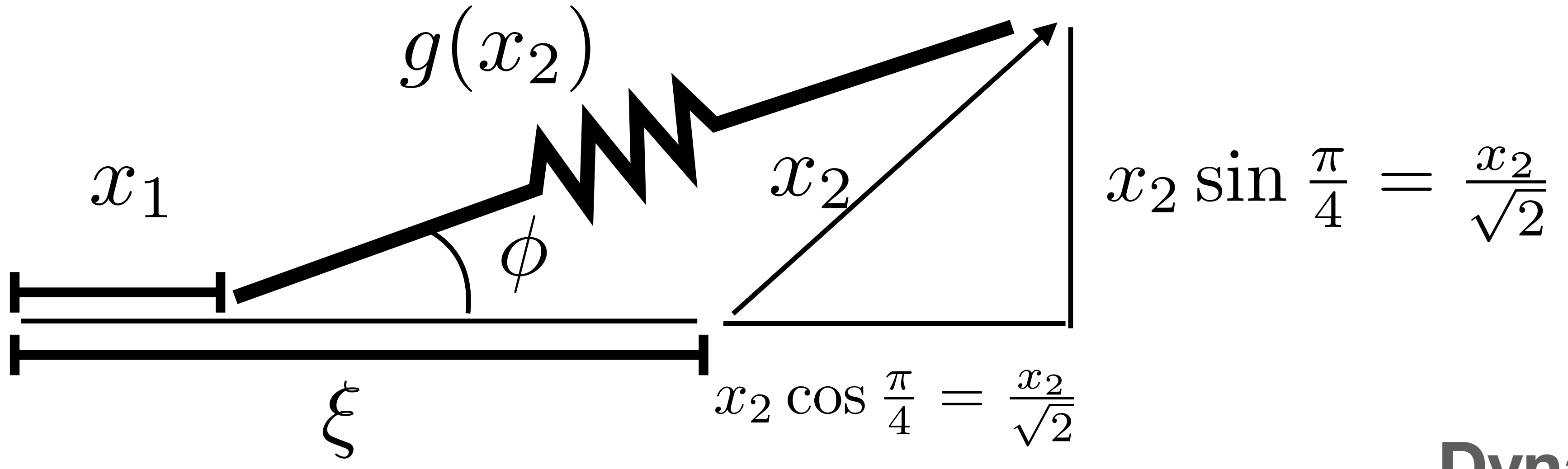


Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Nonlinear spring setup:



Spring length:

$$g(x_1, x_2, \xi) = \left\| \begin{bmatrix} \xi - x_1 + x_2/\sqrt{2} \\ x_2/\sqrt{2} \end{bmatrix} \right\|_2$$

$$= \sqrt{(\xi - x_1 + x_2/\sqrt{2})^2 + x_2^2/2}$$

Spring angle

$$\phi(x_1, x_2, \xi) = \tan^{-1}\left(\frac{x_2}{\sqrt{2}(\xi - x_1) + x_2}\right)$$

Spring force: $F_s = k_2\big(g(x_1, x_2, \xi) - \xi\big)$

Dynamics: $\sum (\text{forces})_x = m\ddot{x}$

Block 1: $m_1\ddot{x}_1 = F_1 + k_2\Big(g(x_1, x_2, \xi) - \xi\Big)\cos(\phi) - k_1 x_1 - d_1\dot{x}_1$

Block 2: $m_2\ddot{x}_2 = F_2 - m_2 g \sin\theta - F_s\cos(\theta - \phi)$

$$= F_2 - m_2 g \frac{1}{\sqrt{2}} - k_2\big(g(x_1, x_2, \xi) - \xi\big)\cos\left(\frac{\pi}{4} - \phi\right)$$

Dynamics

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 + k_2\Big(g(x_1, x_2, \xi) - \xi\Big)\cos\phi - k_1 x_1 - d_1\dot{x}_1 \\ F_2 - \frac{m_2 g}{\sqrt{2}} - k_2\Big(g(x_1, x_2, \xi) - \xi\Big)\cos\left(\frac{\pi}{4} - \phi\right) \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

Example: Two Blocks - Oblique Connection

$g = 10$

Free Body Diagrams

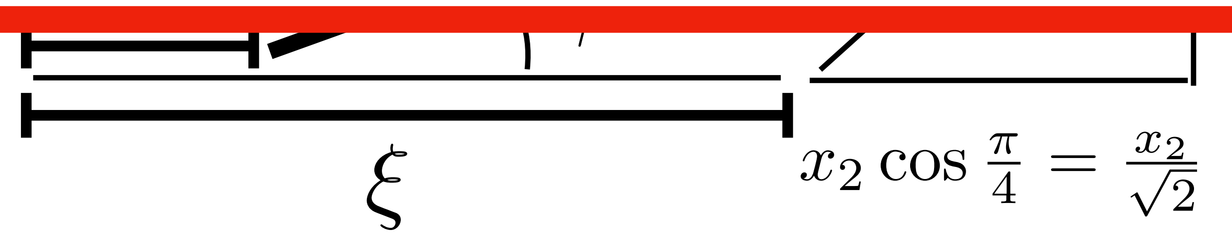
$$\frac{\partial g}{\partial x_1} = \frac{1}{2}\left((\xi - x_1 + x_2/\sqrt{2})^2 + x_2^2/2\right)^{-1/2}(-2)(\xi - x_1 + x_2/\sqrt{2})$$

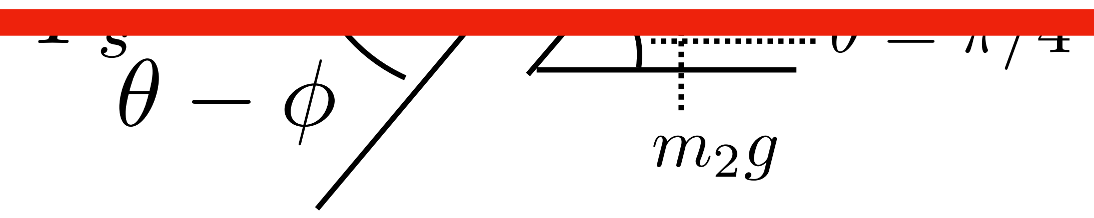
$$\frac{\partial g}{\partial x_2} = \frac{1}{2}\left((\xi - x_1 + x_2/\sqrt{2})^2 + x_2^2/2\right)^{-1/2}\left(\sqrt{2}(\xi - x_1) + 2x_2\right)$$

$$\frac{\partial \phi}{\partial x_1} = \frac{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2+x_2^2}\left(\frac{\sqrt{2}x_2}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2}\right) = \frac{\sqrt{2}x_2}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2+x_2^2}$$

$$\frac{\partial \phi}{\partial x_2} = \frac{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2+x_2^2}\left(\frac{1}{\sqrt{2}(\xi-x_1)+x_2} + \frac{-x_2}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2}\right) = \frac{\sqrt{2}(\xi-x_1)}{\left(\sqrt{2}(\xi-x_1)+x_2\right)^2+x_2^2}$$

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} k_2\frac{\partial g}{\partial x_1}\cos\phi - k_2(g-\xi)\sin\phi\frac{\partial\phi}{\partial x_1} - k_1 & k_2\frac{\partial g}{\partial x_2}\cos\phi - k_2(g-\xi)\sin\phi\frac{\partial\phi}{\partial x_2} & -d_1 & 0 \\ -k_2\frac{\partial g}{\partial x_1}\cos(\frac{\pi}{4}-\phi) - k_2(g-\xi)\sin(\frac{\pi}{4}-\phi)\frac{\partial\phi}{\partial x_1} & -k_2\frac{\partial g}{\partial x_2}\cos(\frac{\pi}{4}-\phi) - k_2(g-\xi)\sin(\frac{\pi}{4}-\phi)\frac{\partial\phi}{\partial x_2} & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$





Dynamics: $\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 + k_2\left(g(x_1,x_2,\xi) - \xi\right)\cos\phi - k_1x_1 - d_1\dot{x}_1 \\ F_2 - \frac{m_2 g}{\sqrt{2}} - k_2\left(g(x_1,x_2,\xi)-\xi\right)\cos\left(\frac{\pi}{4}-\phi\right) \end{bmatrix}}_{\mathbf{F}}$$

Spring length:

$$g(x_1, x_2, \xi) = \left\|\begin{bmatrix} \xi - x_1 + x_2/\sqrt{2} \\ x_2/\sqrt{2} \end{bmatrix}\right\|_2$$

$$= \sqrt{(\xi - x_1 + x_2/\sqrt{2})^2 + x_2^2/2}$$

Linearization $\dot{z} = f(z,u)$ $\quad z = \begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}^T \quad u = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^T$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \multicolumn{2}{c}{\mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u} \end{bmatrix}$$

Spring angle

$$\phi(x_1, x_2, \xi) = \tan^{-1}\left(\frac{x_2}{\sqrt{2}(\xi - x_1) + x_2}\right)$$

Spring force: $F_s = k_2\left(g(x_1, x_2, \xi) - \xi\right)$

$$\frac{\partial \mathbf{a}}{\partial z} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \qquad \frac{\partial \mathbf{a}}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

RLC Circuits - Circuit 1

SECOND ORDER SYSTEMS (2nd order derivatives)

RLC Circuits

… one state per object with dynamics

States:

Capacitor state: charge or voltage ...analogous to position

(proportional)

Inductor state: current ...analogous to velocity

Dynamics:

$$\dot{Q}_C = I_C$$

$$L\dot{I}_L = V_L$$

Graph Relations:

KVL: sum of voltage around a loop = 0

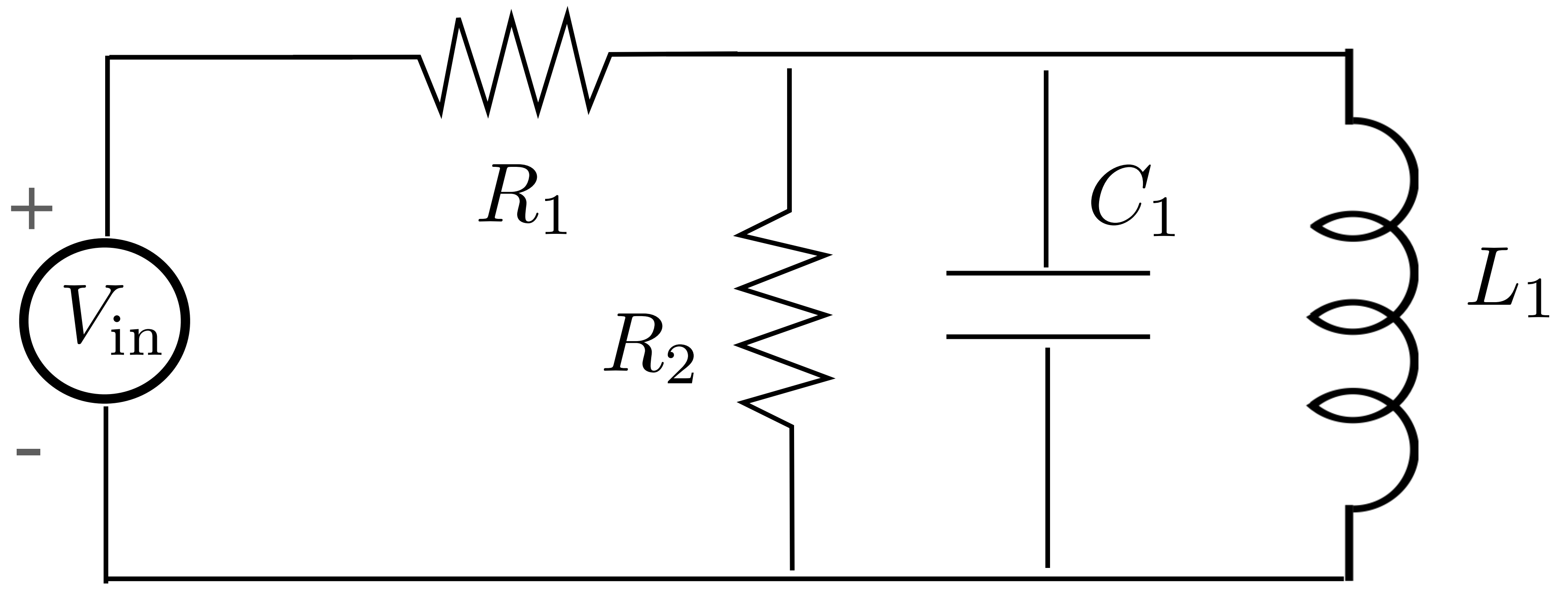
KCL sum of currents into a node = 0

Elements:

$$Q_C = CV_C$$

$$V_R = RI_R$$

$$V_L = L\dot{I}_L$$



States:

Q_{C_1} Charge on capacitor 1

I_{L_1} Current thru inductor 1

KVL/KCL Equations:

KVL $$V_{\text{in}} = V_{R_1} + V_{R_2}$$

$$V_{R_2} = V_{C_1} = V_{L_1}$$

KCL $$I_{R_1} = I_{R_2} + I_{C_1} + I_{L_1}$$

Derivations

Capacitor dynamics (in terms of states…)

$$\dot{Q}_{C_1} = I_{C_1} = I_{R_1} - I_{R_2} - I_{L_1} = \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1} - \frac{Q_{C_1}}{R_2 C_1} - I_{L_1}$$

since…

$$V_{\text{in}} = R_1 I_{R_1} + \frac{Q_{C_1}}{C_1} \quad \Rightarrow \quad I_{R_1} = \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1}$$

$$R_2 I_{R_2} = \frac{Q_{C_1}}{C_1} \quad \Rightarrow \quad I_{R_2} = \frac{Q_{C_1}}{R_2 C_1}$$

Inductor dynamics (in terms of states…)

$$L\dot{I}_{L_1} = V_{L_1} = V_{C_1} = \frac{Q_{C_1}}{C_1}$$

Dynamics

$$\dot{Q}_{C_1} = \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1} - \frac{Q_{C_1}}{R_2 C_1} - I_{L_1} \qquad \dot{I}_{L_1} = \frac{Q_{C_1}}{L_1 C_1}$$

State Space

$$\begin{bmatrix} \dot{Q}_{C_1} \\ \dot{I}_{L_1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} & -1 \\ \frac{1}{L_1 C_1} & 0 \end{bmatrix} \begin{bmatrix} Q_{C_1} \\ I_{L_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} \\ 0 \end{bmatrix} V_{\text{in}}$$

RLC Circuits - Circuit 1

Linearization Stability - Case 1

$$L_1 = 1 \quad C_1 = 1$$

$$R_1 = R_2 = 0.001$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -2000 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = -2000 \qquad \lambda_2 = -0.0005$$

Comments

One mode decays very fast and one mode decays very slowly

Linearization Stability - Case 2

$$L_1 = 1 \quad C_1 = 1$$

$$R_1 = R_2 = 100$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.02 & -1 \\ 1 & 0 \end{bmatrix}$$

Eigenvalues $\lambda_{1,2} = -0.01 \pm 1.00i$

Comments

Due to large resistances, electrical energy tends to oscillate back and forth between the capacitor and inductor

KVL/KCL Equations:

$$= V_{R_1} + V_{R_2}$$

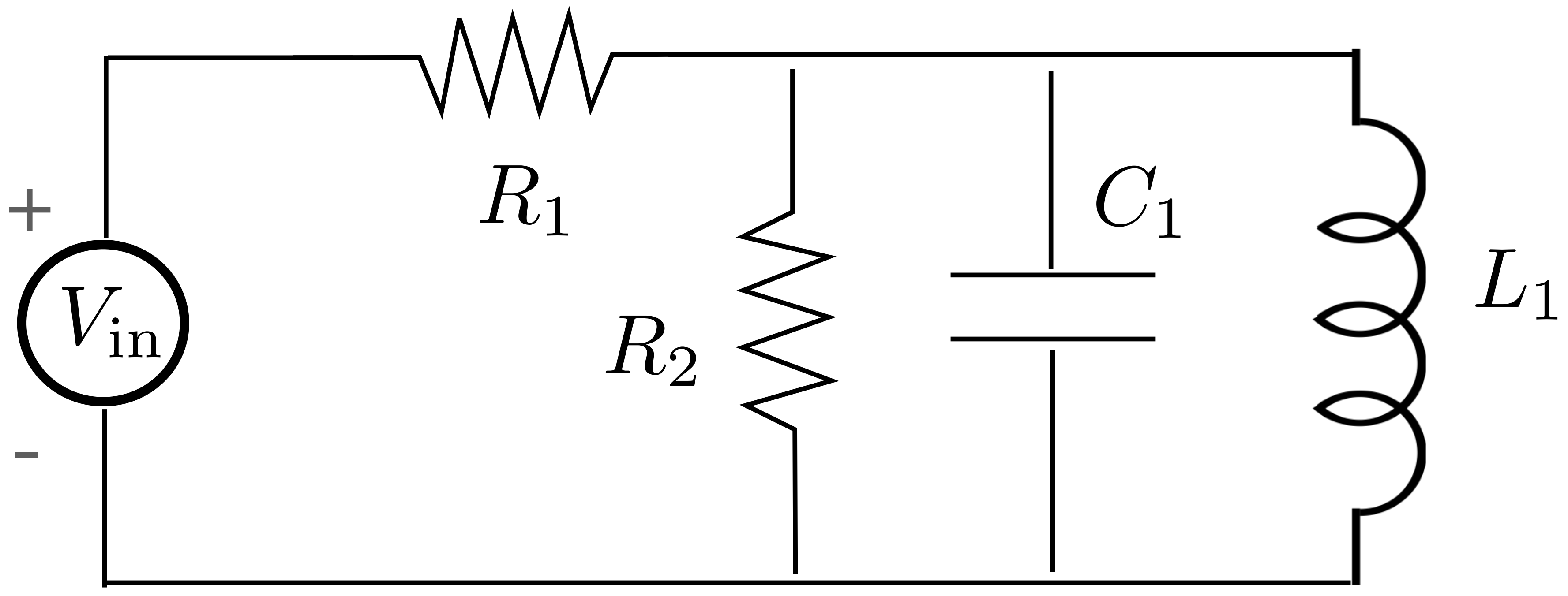
$$= V_{C_1} = V_{L_1}$$

$$= I_{R_2} + I_{C_1} + I_{L_1}$$

$$= \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1} - \frac{Q_{C_1}}{R_2 C_1} - I_{L_1}$$

$$= \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1}$$

$$I_{R_2} = \frac{Q_{C_1}}{R_2 C_1}$$



Dynamics

$$\dot{Q}_{C_1} = \frac{V_{\text{in}}}{R_1} - \frac{Q_{C_1}}{R_1 C_1} - \frac{Q_{C_1}}{R_2 C_1} - I_{L_1} \qquad \dot{I}_{L_1} = \frac{Q_{C_1}}{L_1 C_1}$$

State Space

$$\begin{bmatrix} \dot{Q}_{C_1} \\ \dot{I}_{L_1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} & -1 \\ \frac{1}{L_1 C_1} & 0 \end{bmatrix} \begin{bmatrix} Q_{C_1} \\ I_{L_1} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} \\ 0 \end{bmatrix} V_{\text{in}}$$

RLC Circuits - Circuit 2

SECOND ORDER SYSTEMS (2nd order derivatives)

RLC Circuits

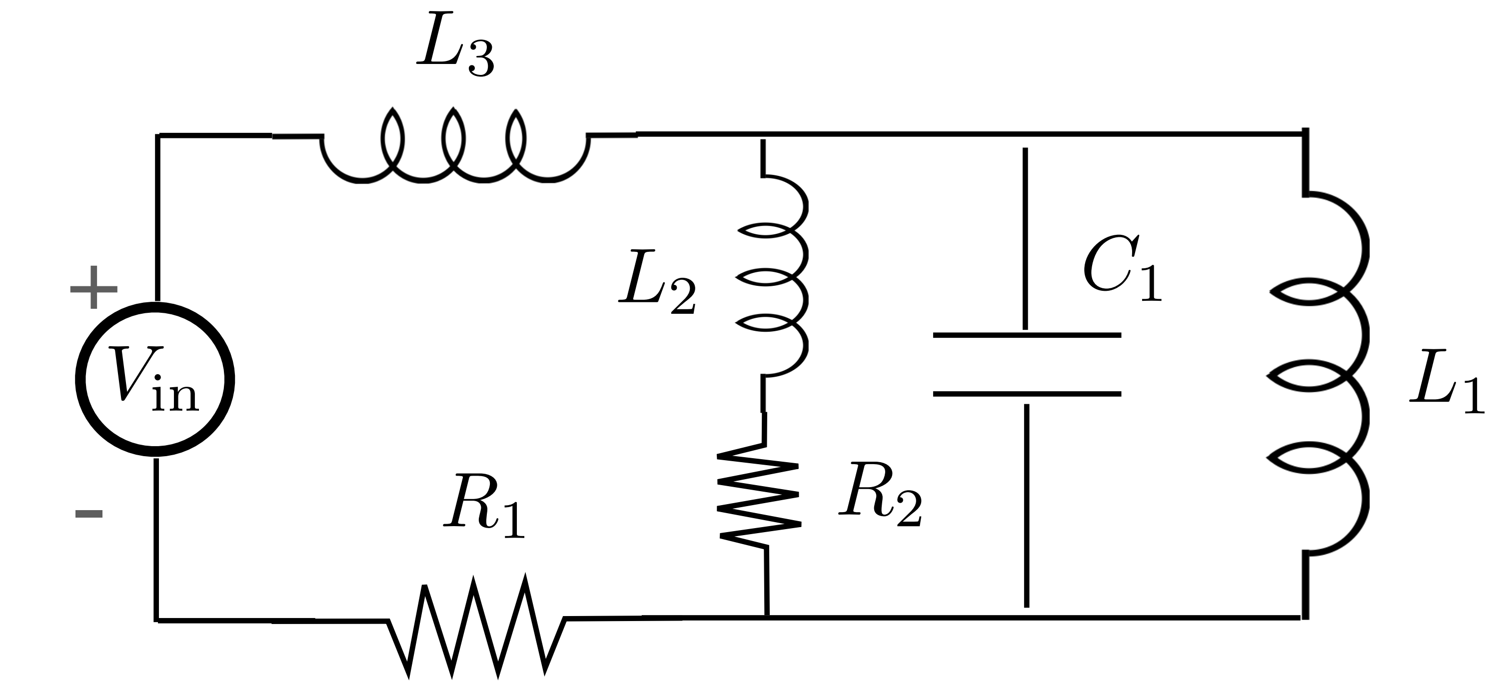
… one state per object with dynamics

States:

Capacitor state: charge or voltage **(proportional)** …analogous to position

Inductor state: current …analogous to velocity

Dynamics:	Graph Relations:		Elements:
$\dot{Q}_C = I_C$	KVL:	sum of voltage around a loop = 0	$Q_C = CV_C$
$L\dot{I}_L = V_L$	KCL	sum of currents into a node = 0	$V_R = RI_R$
			$V_L = L\dot{I}_L$



States:

Q_{C_1}	Charge on capacitor 1
I_{L_1}	Current thru inductor 1
I_{L_2}	Current thru inductor 2
I_{L_3}	Current thru inductor 3

KVL/KCL Equations:

KVL

$$V_{\text{in}} = V_{L_3} + V_{C_1} + V_{R_2}$$
$$V_{R_1} = V_{C_1} = V_{L_2} + V_{R_2}$$

KCL

$$I_{L_3} = I_{R_1} = I_{L_1} + I_{C_1} + I_{L_2}$$
$$I_{L_2} = I_{R_2}$$

Derivations

Capacitor dynamics (in terms of states…)

$$\dot{Q}_{C_1} = I_{C_1} = I_{L_3} - I_{L_1} - I_{L_2}$$

Inductor dynamics (in terms of states…)

$$L_1\dot{I}_{L_1} = V_{L_1} = V_{C_1} = \frac{Q_{C_1}}{C_1} \qquad L_2\dot{I}_{L_2} = V_{L_2} = \frac{Q_{C_1}}{C_1} - R_2 I_{L_2}$$

$$L_3\dot{I}_{L_3} = V_{L_3} = V_{\text{in}} - R_1 I_{L_3} - \frac{Q_{C_1}}{C_1}$$

since…

$$V_{C_1} = V_{L_1} = V_{L_2} = V_{L_2} + V_{R_2} = \frac{Q_{C_1}}{C_1} \qquad I_{L_2} = I_{R_2}$$

Dynamics

$$\dot{Q}_{C_1} = I_{L_3} - I_{L_1} - I_{L_2} \qquad L_2\dot{I}_{L_2} = \frac{Q_{C_1}}{C_1} - R_2 I_{L_2}$$
$$L_1\dot{I}_{L_1} = \frac{Q_{C_1}}{C_1} \qquad L_3\dot{I}_{L_3} = V_{\text{in}} - R_1 I_{L_3} - \frac{Q_{C_1}}{C_1}$$

State Space

$$\begin{bmatrix} \dot{Q}_{C_1} \\ \dot{I}_{L_1} \\ \dot{I}_{L_2} \\ \dot{I}_{L_3} \end{bmatrix} = \begin{bmatrix} 0 & \boxed{-1} & \boxed{-1} & \boxed{1} \\ \frac{1}{L_1C_1} & 0 & 0 & 0 \\ \frac{1}{L_2C_1} & 0 & \frac{-R_2}{L_2} & 0 \\ \frac{-1}{L_3C_1} & 0 & \boxed{0} & \frac{-R_1}{L_3} \end{bmatrix} \begin{bmatrix} Q_{C_1} \\ I_{L_1} \\ I_{L_2} \\ I_{L_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boxed{\frac{1}{L_3}} \end{bmatrix} V_{\text{in}}$$

RLC Circuits - Circuit 2

States: KVL/KCL Equations:

Linearization Stability - Case 1

$$L_1 = L_2 = L_3 = 1 \qquad C_1 = 1$$

$$R_1 = R_2 = 0.001$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -0.001 & 0 \\ -1 & 0 & 0 & -0.001 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = -0.00033 \qquad \lambda_2 = -0.001$$
$$\lambda_{3,4} = -0.00033 \pm 1.732i$$

Comments

Linearization Stability - Case 2

$$L_1 = L_2 = L_3 = 1 \qquad C_1 = 1$$

$$R_1 = R_2 = 100$$

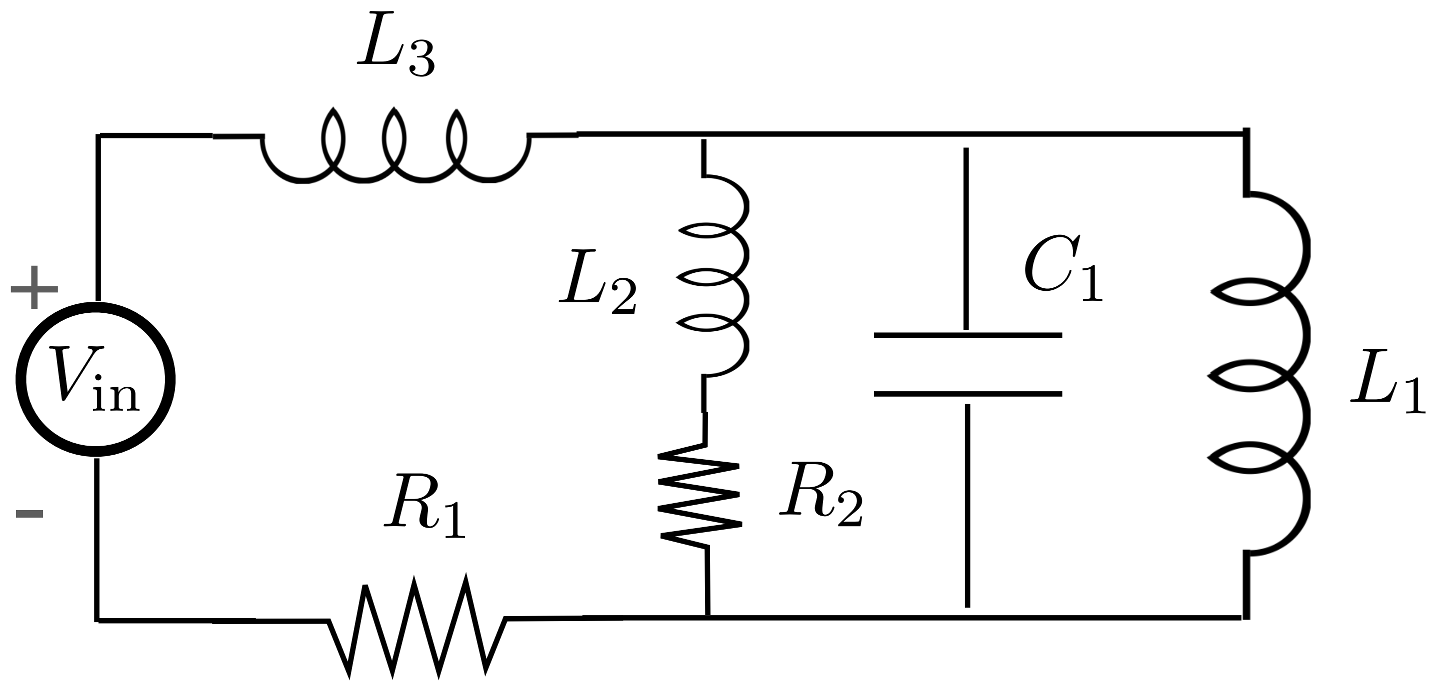
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -100 & 0 \\ -1 & 0 & 0 & -100 \end{bmatrix}$$

Eigenvalues

$$\lambda_{1,2} = -100.00, \;\; \lambda_{3,4} = -0.01 \pm 1.00i$$

Comments

$$= V_{L_3} + V_{C_1} + V_{R_2}$$
$$= V_{C_1} = V_{L_2} + V_{R_2}$$
$$= I_{R_1} = I_{L_1} + I_{C_1} + I_{L_2}$$
$$= I_{R_2}$$
$$- I_{L_2}$$
$$L_2 \dot{I}_{L_2} = V_{L_2} = \frac{Q_{C_1}}{C_1} - R_2 I_{L_2}$$
$$\frac{Q_{C_1}}{C_1}$$
$$= \frac{Q_{C_1}}{C_1} \qquad I_{L_2} = I_{R_2}$$



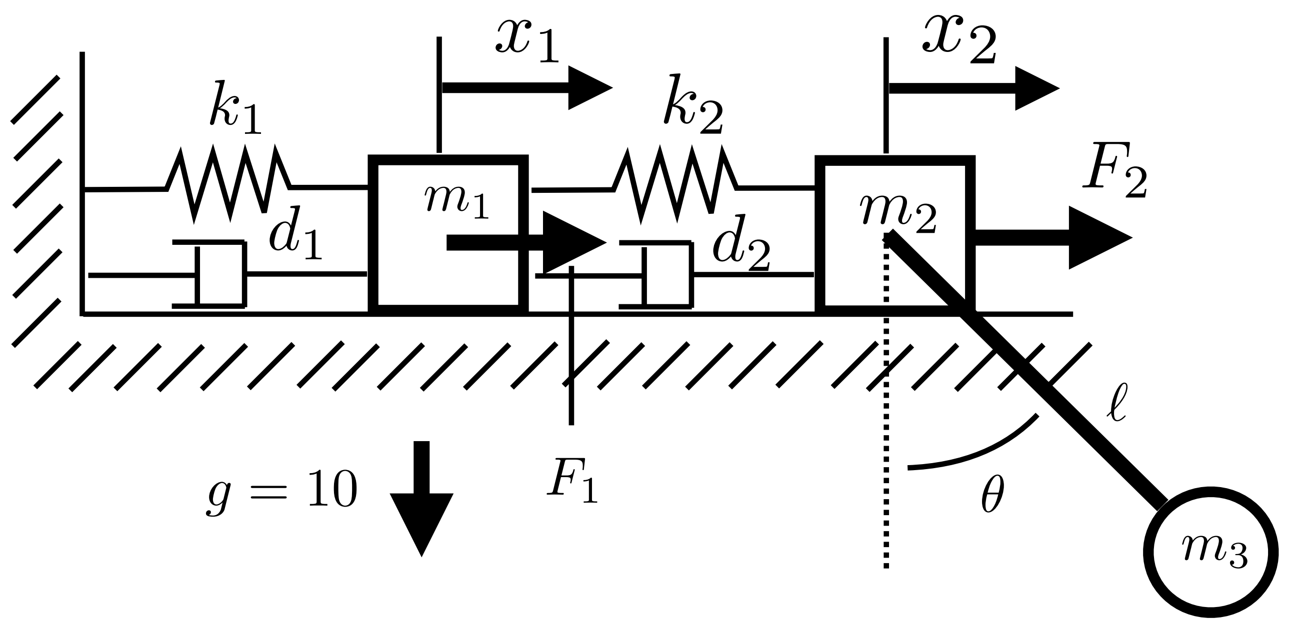
Dynamics

$$\dot{Q}_{C_1} = I_{L_3} - I_{L_1} - I_{L_2} \qquad L_2 \dot{I}_{L_2} = \frac{Q_{C_1}}{C_1} - R_2 I_{L_2}$$
$$L_1 \dot{I}_{L_1} = \frac{Q_{C_1}}{C_1} \qquad L_3 \dot{I}_{L_3} = V_{\text{in}} - R_1 I_{L_3} - \frac{Q_{C_1}}{C_1}$$

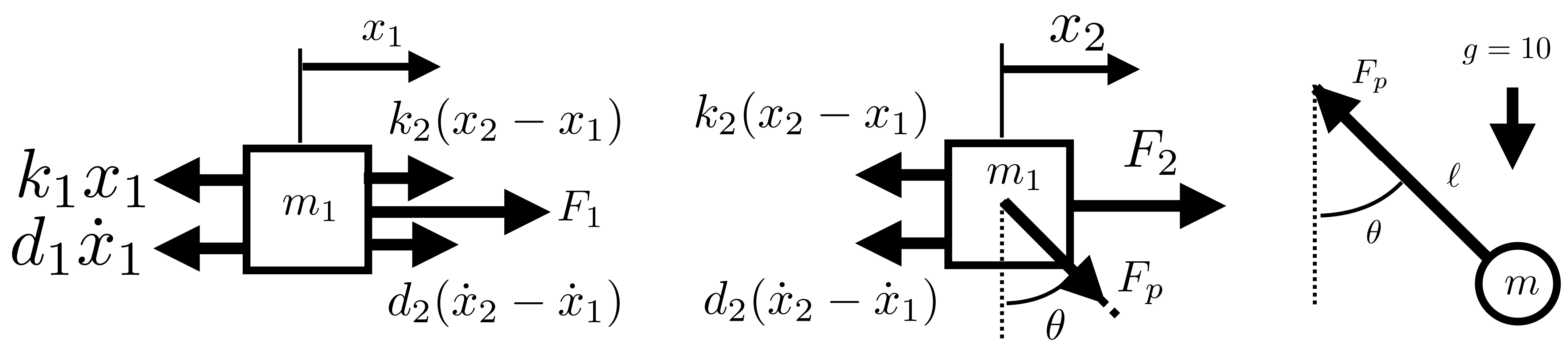
State Space

$$\begin{bmatrix} \dot{Q}_{C_1} \\ \dot{I}_{L_1} \\ \dot{I}_{L_2} \\ \dot{I}_{L_3} \end{bmatrix} = \begin{bmatrix} 0 & \boxed{-1} & \boxed{-1} & \boxed{1} \\ \frac{1}{L_1 C_1} & 0 & 0 & 0 \\ \frac{1}{L_2 C_1} & 0 & \frac{-R_2}{L_2} & 0 \\ \frac{-1}{L_3 C_1} & 0 & \boxed{0} & \frac{-R_1}{L_3} \end{bmatrix} \begin{bmatrix} Q_{C_1} \\ I_{L_1} \\ I_{L_2} \\ I_{L_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \boxed{\frac{1}{L_3}} \end{bmatrix} V_{\text{in}}$$

Example: Two Blocks + Pendulum



Free Body Diagrams



Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Pendulum

$$x_p = x_2 + \ell \sin\theta$$
$$\dot{x}_p = \dot{x}_2 + \dot{\theta}\ell \cos\theta$$
$$\ddot{x}_p = \ddot{x}_2 + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta$$

$$y_p = -\ell\cos\theta$$
$$\dot{y}_p = \dot{\theta}\ell\sin\theta$$
$$\ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$$

Dynamics:

Block 1: $\sum(\text{forces})_x = m\ddot{x}$ ① $m_1\ddot{x}_1 = F_1 - k_1x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1)$

Block 2: $\sum(\text{forces})_x = m\ddot{x}$ ② $m_2\ddot{x}_2 = F_2 + F_p\sin(\theta) - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1)$

Pendulum: $\sum(\text{forces})_x = m\ddot{x}$ ③ $m_3\ddot{x}_p = m_3\ddot{x}_2 + m_3\ddot{\theta}\ell\cos\theta - m_3\dot{\theta}^2\ell\sin\theta = -F_p\sin\theta$

$\sum(\text{forces})_y = m\ddot{y}$ ④ $m_3\ddot{y}_p = m_3\ddot{\theta}\ell\sin\theta + m_3\dot{\theta}^2\ell\cos\theta = F_p\cos\theta - mg$

---

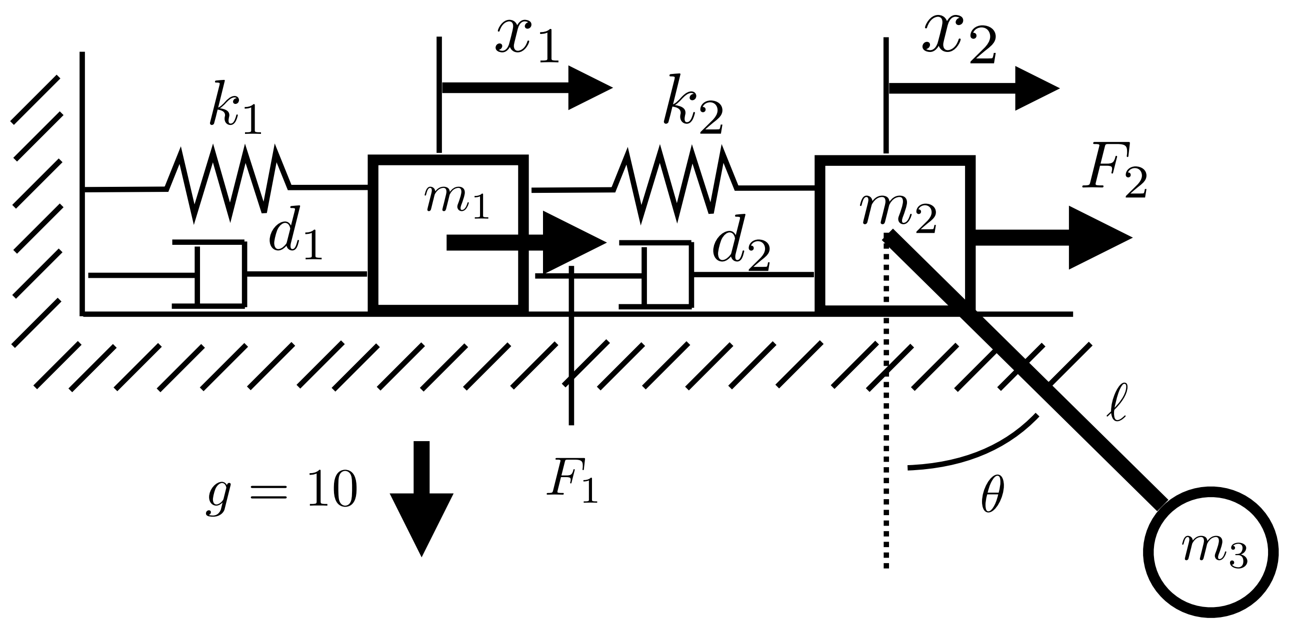
② + ③ $(m_2 + m_3)\ddot{x}_2 + m_3\ddot{\theta}\ell\cos\theta - m_3\dot{\theta}^2\ell\sin\theta = F_2 - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1)$

$\cos\theta$ ③ $+ \sin\theta$ ④ $m_3\ddot{x}_2\cos\theta + m_3\ddot{\theta}\ell = -mg\sin\theta$

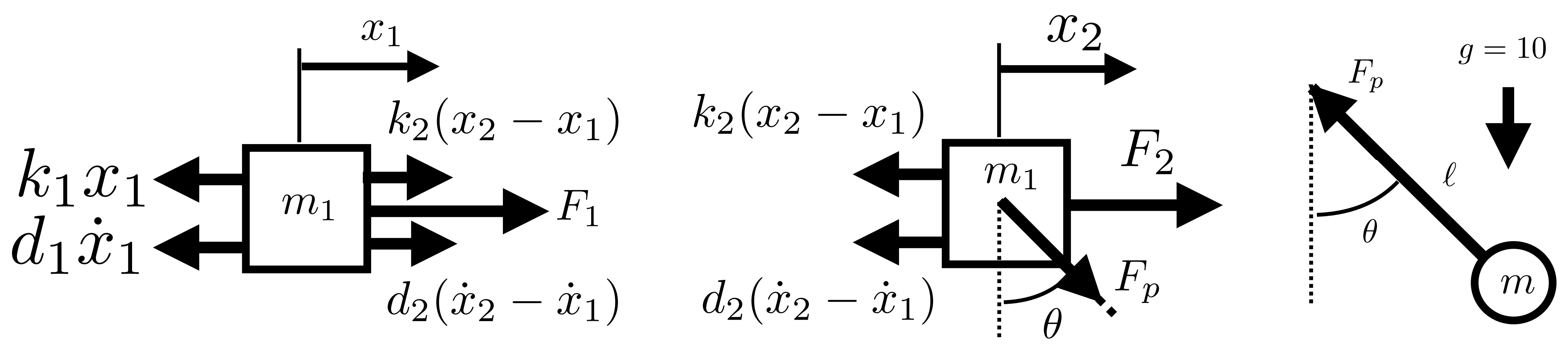
…combining

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2 + m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 - k_1x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1) \\ F_2 + m_3\dot{\theta}^2\ell\sin\theta - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1) \\ -mg\ell\sin\theta \end{bmatrix}}_{\mathbf{F}}$$

Example: Two Blocks + Pendulum



Free Body Diagrams



Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Pendulum

$$x_p = x_2 + \ell \sin\theta$$

$$\dot{x}_p = \dot{x}_2 + \dot{\theta}\ell\cos\theta$$

$$\ddot{x}_p = \ddot{x}_2 + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta$$

$$y_p = -\ell\cos\theta$$

$$\dot{y}_p = \dot{\theta}\ell\sin\theta$$

$$\ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$$

Dynamics:

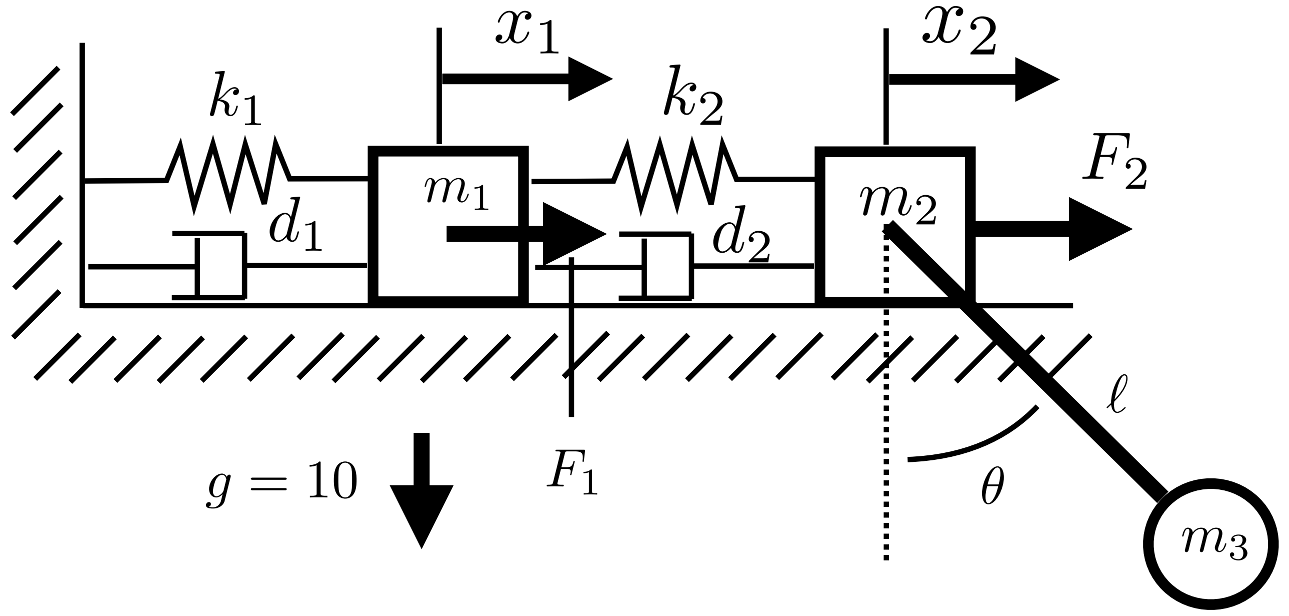
$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 - k_1 x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1) \\ F_2 + m_3\dot{\theta}^2\ell\sin\theta - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1) \\ -m_3 g\ell\sin\theta \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

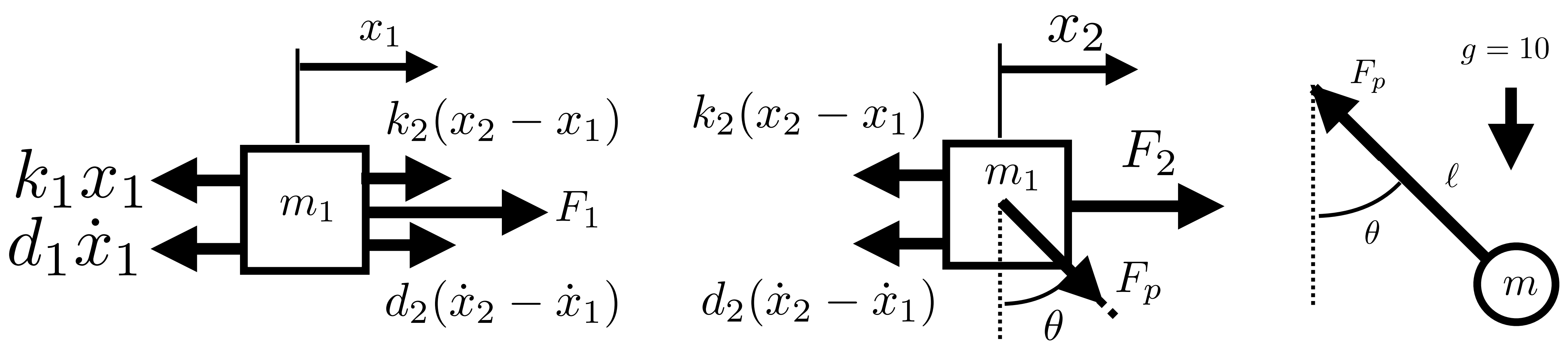
$$\dot{z} = f(z, u) \qquad z = [\ x_1\ x_2\ \theta\ \dot{x}_1\ \dot{x}_2\ \dot{\theta}\ ]^T \qquad u = [\ F_1\ F_2\ ]^T$$

$$\dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{bmatrix} \\ \begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}^{-1} \begin{bmatrix} F_1 - k_1 x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1) \\ F_2 + m_3\dot{\theta}^2\ell\sin\theta - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1) \\ -mg\ell\sin\theta \end{bmatrix} \end{bmatrix}$$

Example: Two Blocks + Pendulum



Free Body Diagrams



Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Pendulum

$$x_p = x_2 + \ell \sin\theta$$

$$\dot{x}_p = \dot{x}_2 + \dot{\theta}\ell\cos\theta$$

$$\ddot{x}_p = \ddot{x}_2 + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta$$

$$y_p = -\ell\cos\theta$$

$$\dot{y}_p = \dot{\theta}\ell\sin\theta$$

$$\ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$$

Dynamics:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 - k_1 x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1) \\ F_2 + m_3\dot{\theta}^2\ell\sin\theta - k_2(x_2 - x_1) - d_2(\dot{x}_2 - \dot{x}_1) \\ -m_3 g\ell\sin\theta \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

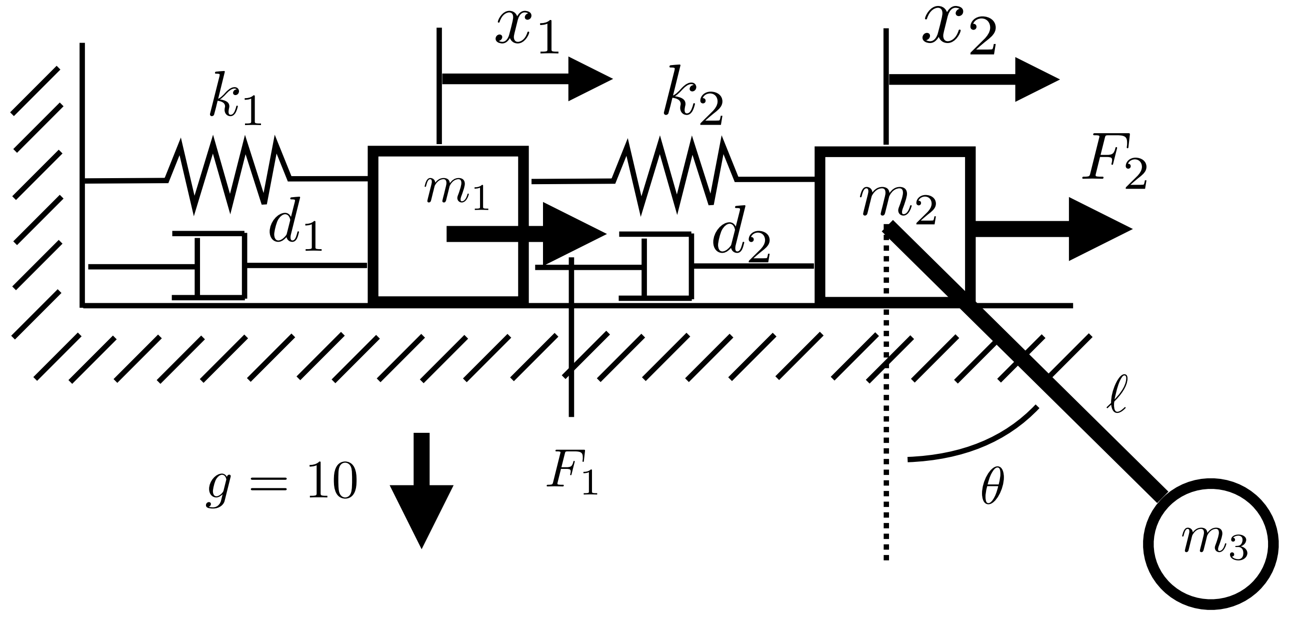
$$\dot{z} = f(z, u) \qquad z = [\ x_1\ x_2\ \theta\ \dot{x}_1\ \dot{x}_2\ \dot{\theta}\ ]^T \qquad u = [\ F_1\ F_2\ ]^T$$

Linearization:

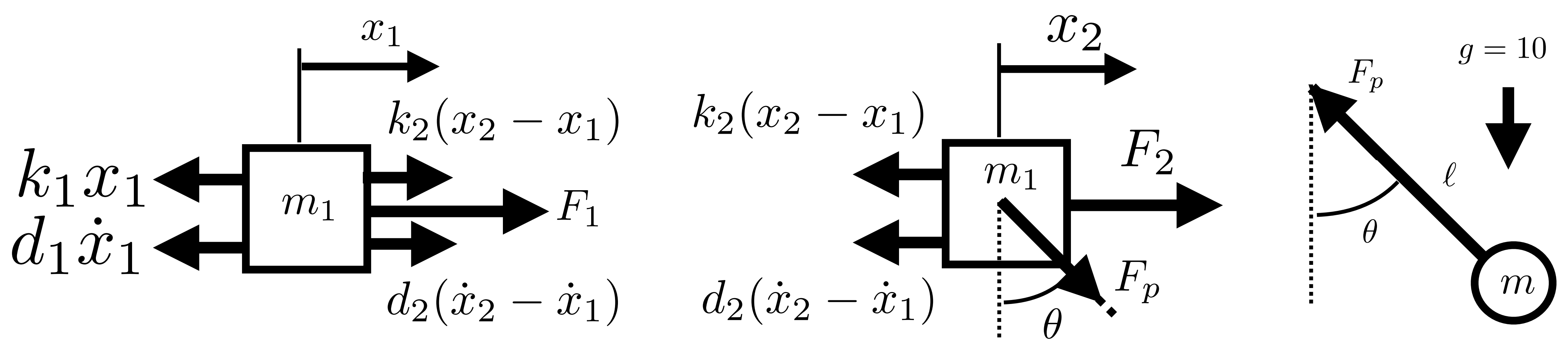
$$\frac{\partial \mathbf{a}}{\partial z} = \left[\frac{\partial \mathbf{M}^{-1}}{\partial x_1}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial x_2}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \theta}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{x}_1}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{x}_2}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}}\mathbf{F} \right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

$$= -\left[\mathbf{M}^{-1}\overset{0}{\cancel{\tfrac{\partial \mathbf{M}}{\partial x_1}}}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\tfrac{\partial \mathbf{M}}{\partial x_2}}}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\tfrac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\tfrac{\partial \mathbf{M}}{\partial \dot{x}_1}}}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\tfrac{\partial \mathbf{M}}{\partial \dot{x}_2}}}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\tfrac{\partial \mathbf{M}}{\partial \dot{\theta}}}}\mathbf{M}^{-1}\mathbf{F} \right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

Example: Two Blocks + Pendulum



Free Body Diagrams



Kinematics

Block 1 $x_1, \dot{x}_1, \ddot{x}_1$

Block 2 $x_2, \dot{x}_2, \ddot{x}_2$

Pendulum

$$x_p = x_2 + \ell \sin\theta$$
$$\dot{x}_p = \dot{x}_2 + \dot{\theta}\ell\cos\theta$$
$$\ddot{x}_p = \ddot{x}_2 + \ddot{\theta}\ell\cos\theta - \dot{\theta}^2\ell\sin\theta$$

$$y_p = -\ell\cos\theta$$
$$\dot{y}_p = \dot{\theta}\ell\sin\theta$$
$$\ddot{y}_p = \ddot{\theta}\ell\sin\theta + \dot{\theta}^2\ell\cos\theta$$

Dynamics:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} F_1 - k_1 x_1 - d_1\dot{x}_1 + k_2(x_2 - x_1) + d_2(\dot{x}_2 - \dot{x}_1) \\ F_2 + m_3\dot{\theta}^2\ell\sin\theta - k_2(x_2-x_1) - d_2(\dot{x}_2 - \dot{x}_1) \\ -m_3 g\ell\sin\theta \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

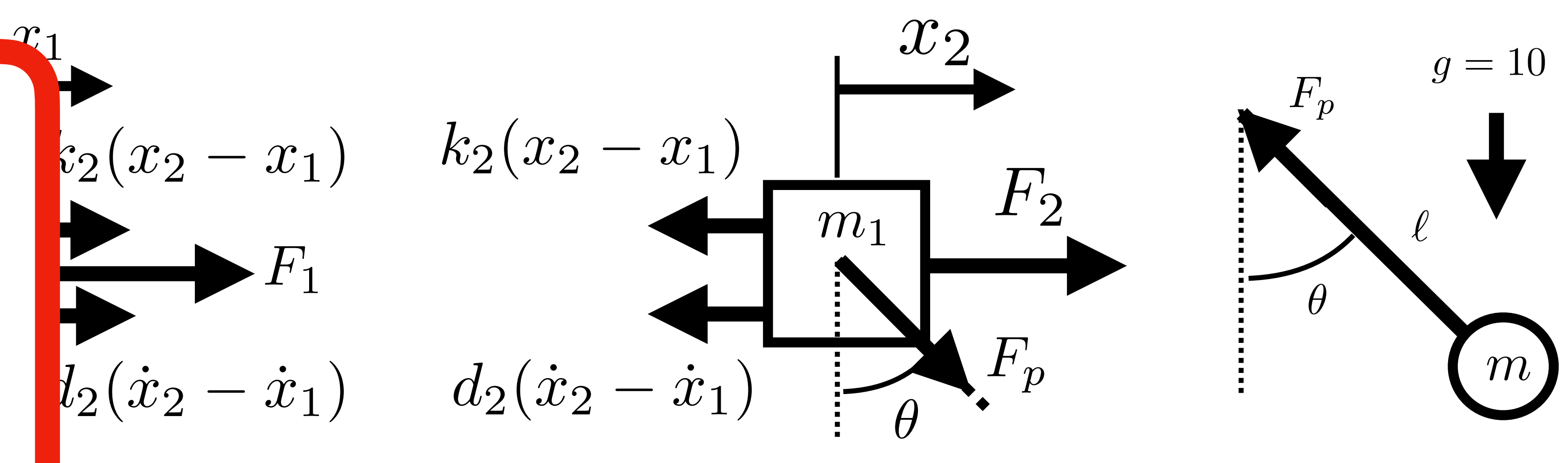
$$\dot{z} = f(z,u) \qquad z = [\ x_1\ \ x_2\ \ \theta\ \ \dot{x}_1\ \ \dot{x}_2\ \ \dot{\theta}\ ]^T \qquad u = [\ F_1\ \ F_2\ ]^T$$

Linearization:

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \multicolumn{2}{c}{\begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial\theta}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Example: Two Blocks + Pendulum

Free Body Diagrams



$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k_1-k_2 & k_2 & 0 & -d_1-d_2 & d_2 & 0 \\ k_2 & -k_2 & m_3\dot\theta^2\ell\cos\theta & d_2 & -d_2 & 2\dot\theta m_3\ell\sin\theta \\ 0 & 0 & -m_3 g\ell\cos\theta & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\frac{1}{\det}\begin{bmatrix} m_3\ell^2 & -m_3\ell\cos\theta \\ -m_3\ell\cos\theta & m_2+m_3 \end{bmatrix}} \end{bmatrix}$$

$$\det = m_3\ell^2(m_2+m_3) + m_3^2\ell\cos\theta$$

ics:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix} \begin{bmatrix} \ddot x_1 \\ \ddot x_2 \\ \ddot\theta \end{bmatrix} = \begin{bmatrix} F_1 - k_1x_1 - d_1\dot x_1 + k_2(x_2-x_1) + d_2(\dot x_2 - \dot x_1) \\ F_2 + m_3\dot\theta^2\ell\sin\theta - k_2(x_2-x_1) - d_2(\dot x_2-\dot x_1) \\ -m_3 g\ell\sin\theta \end{bmatrix}$$

$\mathbf{M}$ $\quad$ $\mathbf{a}$ $\quad$ $\mathbf{F}$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$y_p = -\ell\cos\theta$

$$\dot z = f(z,u) \qquad z = [\ x_1\ \ x_2\ \ \theta\ \ \dot x_1\ \ \dot x_2\ \ \dot\theta\ ]^T \qquad u = [\ F_1\ \ F_2\ ]^T$$

$$\frac{\partial \mathbf{M}}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -m_3\ell\sin\theta \\ 0 & -m_3\ell\sin\theta & 0 \end{bmatrix}$$

ization:

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \begin{bmatrix} 0 & 0 & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial\theta}\mathbf{M}^{-1}\mathbf{F} & 0 & 0 & 0 \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} & \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Example: Two Blocks + Pendulum

Free Body Diagrams

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k_1 - k_2 & k_2 & 0 & -d_1 - d_2 & d_2 & 0 \\ k_2 & -k_2 & m_3 \dot{\theta}^2 \ell \cos\theta & d_2 & -d_2 & 2\dot{\theta} m_3 \ell \sin\theta \\ 0 & 0 & -m_3 g \ell \cos\theta & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\frac{1}{\det}\begin{bmatrix} m_3\ell^2 & -m_3\ell\cos\theta \\ -m_3\ell\cos\theta & m_2 + m_3 \end{bmatrix}} \\ 0 & & \end{bmatrix}$$

$$\det = m_3 \ell^2 (m_2 + m_3) + m_3^2 \ell \cos\theta$$

$$y_p = -\ell\cos\theta$$

$$\frac{\partial \mathbf{M}}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -m_3 \ell \sin\theta \\ 0 & -m_3 \ell \sin\theta & 0 \end{bmatrix}$$

ics:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2 + m_3) & m_3 \ell \cos\theta \\ 0 & m_3 \ell \cos\theta & m_3 \ell^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F_1 - \\ F_2 + \\ \end{bmatrix}$$

$$\mathbf{M} \qquad \mathbf{a}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\dot{z} = f(z, u) \qquad z = [\, x_1 \; x_2 \; \theta \; \dot{x}$$

ization:

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} & \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Linearization Stability - Case 1

$m_1 = m_2 = m_3 = 1$

$k_1 = k_2 = 1$

$d_1 = d_2 = 1$

$\theta_1 = 0$

$\ell = 1$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. \\ -2. & 1. & 0. & -2. & 1. & 0. \\ 1. & -1. & 10. & 1. & -1. & 0. \\ -1. & 1. & -20. & -1. & 1. & 0. \end{bmatrix}$$

Eigenvalues

$\lambda_{1,2} = -0.245 \pm 4.345i \quad \lambda_{5,6} = -0.108 \pm 0.453i$

$\lambda_{3,4} = -1.147 \pm 1.007i$

Example: Two Blocks + Pendulum

Free Body Diagrams

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k_1 - k_2 & k_2 & 0 & -d_1 - d_2 & d_2 & 0 \\ k_2 & -k_2 & m_3\dot{\theta}^2 \ell \cos\theta & d_2 & -d_2 & 2\dot{\theta} m_3 \ell \sin\theta \\ 0 & 0 & -m_3 g \ell \cos\theta & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\frac{1}{\det}\begin{bmatrix} m_3\ell^2 & -m_3\ell\cos\theta \\ -m_3\ell\cos\theta & m_2 + m_3 \end{bmatrix}} \\ 0 & & \end{bmatrix}$$

ics:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2 + m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F_1 - \\ F_2 + \end{bmatrix}$$

$\mathbf{M}$ $\qquad$ $\mathbf{a}$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\dot{z} = f(z, u) \qquad z = [\, x_1 \; x_2 \; \theta \; \dot{x}$$

on:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ [\mathbf{0} \;\; \mathbf{0} \;\; -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} \;\; \mathbf{0} \;\; \mathbf{0} \;\; \mathbf{0}] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} & \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Comments

Notice that the imaginary parts of the evals are smaller than when the pendulum length was shorter. Increasing pendulum length leads to slower oscillations of the system.

The real parts of the evals have slightly smaller magnitudes meaning the system decays slower.

Linearization Stability - Case 2

$m_1 = m_2 = m_3 = 1$

$k_1 = k_2 = 1$

$d_1 = d_2 = 1$

$\theta_1 = 0$

$\ell = 100$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0. & 0. & 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 0. & 1. & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. \\ -2. & 1. & 0. & -2. & 1. & 0. \\ 1. & -1. & 10. & 1. & -1. & 0. \\ -0.01 & 0.01 & -0.2 & -0.01 & 0.01 & 0. \end{bmatrix}$$

Eigenvalues

$\lambda_{1,2} = -1.31 \pm 0.97i$ $\qquad$ $\lambda_{5,6} = -0.0091 \pm 0.285i$

$\lambda_{3,4} = -0.18 \pm 0.66i$

Example: Two Blocks + Pendulum

Free Body Diagrams

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k_1 - k_2 & k_2 & 0 & -d_1 - d_2 & d_2 & 0 \\ k_2 & -k_2 & m_3 \dot{\theta}^2 \ell \cos\theta & d_2 & -d_2 & 2\dot{\theta} m_3 \ell \sin\theta \\ 0 & 0 & -m_3 g \ell \cos\theta & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{\det}\begin{bmatrix} m_3\ell^2 & -m_3\ell\cos\theta \\ -m_3\ell\cos\theta & m_2 + m_3 \end{bmatrix} & \\ 0 & & \end{bmatrix}$$

Linearization Stability - Case 3

$m_1 = m_2 = m_3 = 1$

$k_1 = k_2 = 1$

$d_1 = d_2 = 1$

$\theta = \pi$

$\ell = 1$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0. & 0. & 1. & 0. & 0 \\ 0 & 0. & 0. & 0. & 1. & 0 \\ 0 & 0. & 0. & 0. & 0. & 1. \\ -2. & 1. & 0. & -2. & 1. & 0. \\ 1. & -1. & 10. & 1. & -1. & 0. \\ 1. & -1. & 20. & 1. & -1. & 0. \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 4.25 \qquad \lambda_2 = -4.76$

$\lambda_{3,4} = -1.137 \pm 0.973i$

$\lambda_{5,6} = -0.111 \pm 0.457i$

ics:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2 + m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F_1 - \\ F_2 + \\ \end{bmatrix}$$

$\mathbf{M}$ $\qquad$ $\mathbf{a}$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\dot{z} = f(z, u) \qquad z = [\, x_1 \;\; x_2 \;\; \theta \;\; \dot{x}$$

Comments

Since the pendulum is vertical, there is an unstable mode with a positive real eigenvalue

on:

$$= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} & \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Example: Two Blocks + Pendulum

Free Body Diagrams

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -k_1-k_2 & k_2 & 0 & -d_1-d_2 & d_2 & 0 \\ k_2 & -k_2 & m_3\dot{\theta}^2\ell\cos\theta & d_2 & -d_2 & 2\dot{\theta}m_3\ell\sin\theta \\ 0 & 0 & -m_3 g\ell\cos\theta & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial \mathbf{F}}{\partial u} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{\det}\begin{bmatrix} m_3\ell^2 & -m_3\ell\cos\theta \\ -m_3\ell\cos\theta & m_2+m_3 \end{bmatrix} \\ 0 & & \end{bmatrix}$$

ics:

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & (m_2+m_3) & m_3\ell\cos\theta \\ 0 & m_3\ell\cos\theta & m_3\ell^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta} \end{bmatrix}}_{\mathbf{a}} = \begin{bmatrix} F_1 - \\ F_2 + \end{bmatrix}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\dot{z} = f(z, u) \qquad z = [\ x_1\ x_2\ \theta\ \dot{x}$$

on:

$$= \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \end{bmatrix} \qquad \frac{\partial f}{\partial u} = \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial u}$$

Comments

Since the pendulum is vertical, there is an unstable mode with a positive real eigenvalue

Linearization Stability - Case 4

$m_1 = m_2 = m_3 = 1$

$k_1 = k_2 = 1$

$d_1 = d_2 = 1$

$\theta = \pi$

$\ell = 100$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0. & 0. & 1. & 0. & 0 \\ 0 & 0. & 0. & 0. & 1. & 0 \\ 0 & 0. & 0. & 0. & 0. & 1. \\ -2. & 1. & 0. & -2. & 1. & 0. \\ 1. & -1. & 10. & 1. & -1. & 0. \\ 0.01 & -0.01 & 0.2 & 0.01 & -0.01. & 0. \end{bmatrix}$$

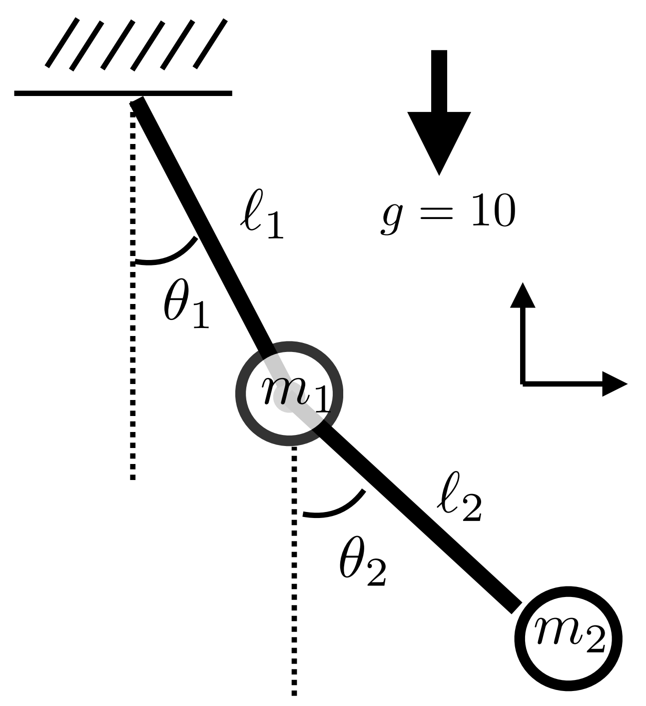
Eigenvalues

$\lambda_{1,2} = -1.308 \pm 0.936i$

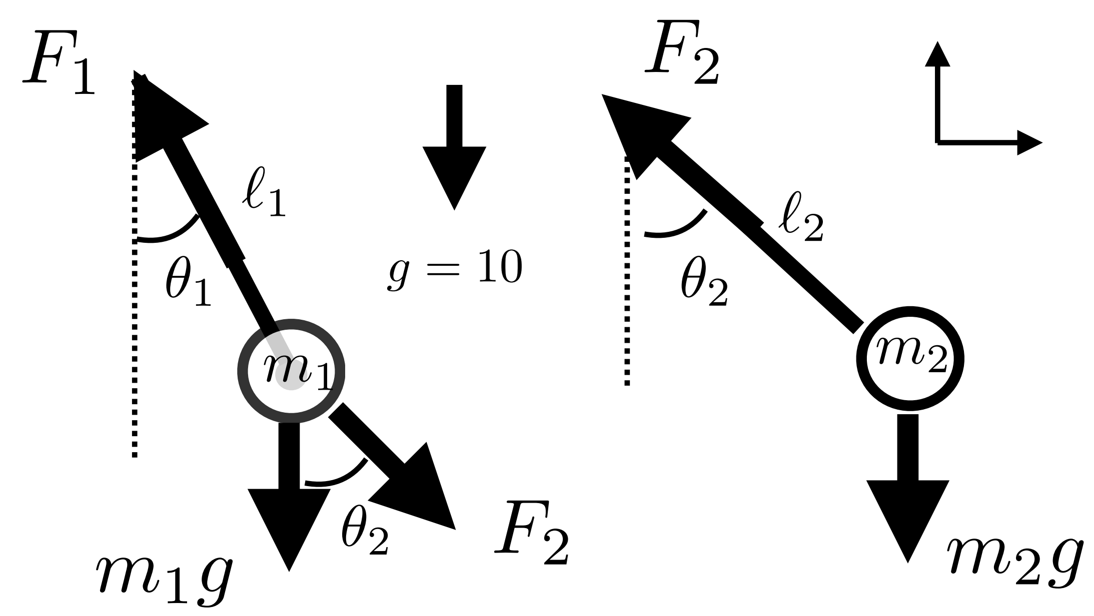
$\lambda_{3,4} = -0.183 \pm 0.537i$

$\lambda_5 = 0.338 \quad \lambda_6 = -0.356$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$

$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$

$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$

$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$

$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Pendulum 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$

$$\dot{x}_2 = \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2$$

$$\ddot{x}_2 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1 + \ddot{\theta}_2 \ell_2 \cos\theta_2 - \dot{\theta}_2^2 \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$

$$\dot{y}_2 = \dot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_2 \ell_2 \sin\theta_2$$

$$\ddot{y}_2 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1 + \ddot{\theta}_2 \ell_2 \sin\theta_2 + \dot{\theta}_2^2 \ell_2 \cos\theta_2$$

Dynamics:

$$\sum (\text{forces})_x = m\ddot{x} \qquad \sum (\text{forces})_y = m\ddot{y}$$

Pendulum 1:

$$(1)\quad -F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1 \qquad (2)\quad F_1 \cos\theta_1 - F_2 \cos\theta_2 - m_1 g = m_1 \ddot{y}_1$$

Pendulum 2:

$$(3)\quad -F_2 \sin\theta_2 = m_2 \ddot{x}_2 \qquad (4)\quad F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$$

Solving to get rid of internal reaction forces... F_1, F_2

$$\frac{1}{\sin\theta_2}(3) + \frac{1}{\cos\theta_2}(4)$$

$$\boxed{0 = \frac{1}{\sin\theta_2} m_2 \ddot{x}_2 + \frac{1}{\cos\theta_2}\left(m_2 g + m_2 \ddot{y}_2\right) \quad (5)}$$

$$(1) + (3)$$

$$-F_1 \sin\theta_1 = m_1 \ddot{x}_1 + m_2 \ddot{x}_2 \quad (6)$$

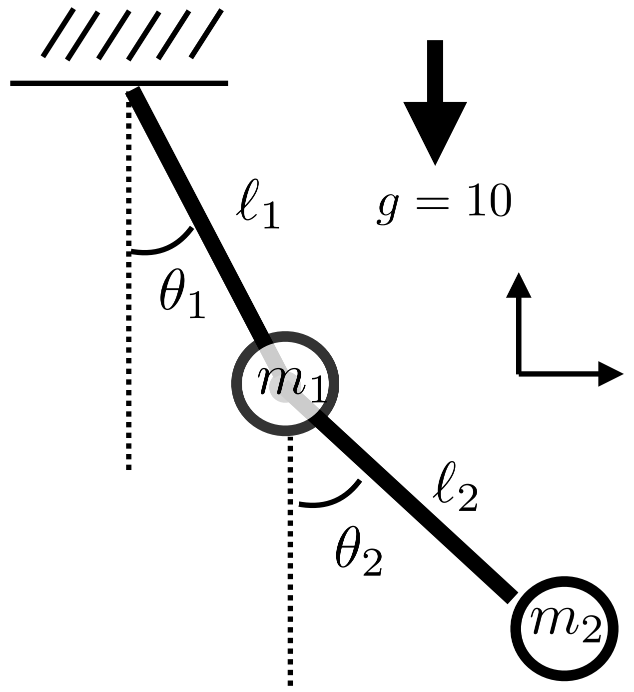
$$(2) + (4)$$

$$F_1 \cos\theta_1 - m_2 g - m_1 g = m_1 \ddot{y}_1 + m_2 \ddot{y}_2 \quad (7)$$

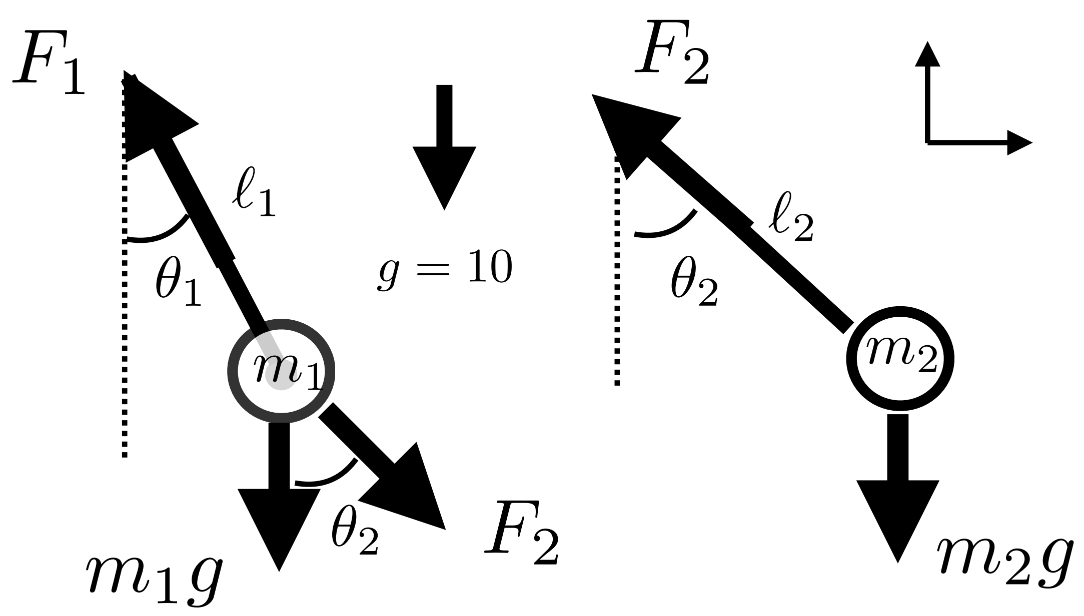
$$\frac{1}{\sin\theta_1}(6) + \frac{1}{\cos\theta_1}(7)$$

$$\boxed{0 = \frac{1}{\sin\theta_1}(m_1 \ddot{x}_1 + m_2 \ddot{x}_2) + \frac{1}{\cos\theta_1}(m_1 g + m_2 g + m_1 \ddot{y}_1 + m_2 \ddot{y}_2) \quad (8)}$$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$
$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$
$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$
$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$
$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Pendulum 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$
$$\dot{x}_2 = \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2$$
$$\ddot{x}_2 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1 + \ddot{\theta}_2 \ell_2 \cos\theta_2 - \dot{\theta}_2^2 \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$
$$\dot{y}_2 = \dot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_2 \ell_2 \sin\theta_2$$
$$\ddot{y}_2 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1 + \ddot{\theta}_2 \ell_2 \sin\theta_2 + \dot{\theta}_2^2 \ell_2 \cos\theta_2$$

Dynamics:

$$\sum (\text{forces})_x = m\ddot{x} \qquad \sum (\text{forces})_y = m\ddot{y}$$

Pendulum 1:

1 $-F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1$

2 $F_1 \cos\theta_1 - F_2 \cos\theta_2 - m_1 g = m_1 \ddot{y}_1$

Pendulum 2:

3 $-F_2 \sin\theta_2 = m_2 \ddot{x}_2$

4 $F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$

Writing in terms of pendulum angles… and simplifying…

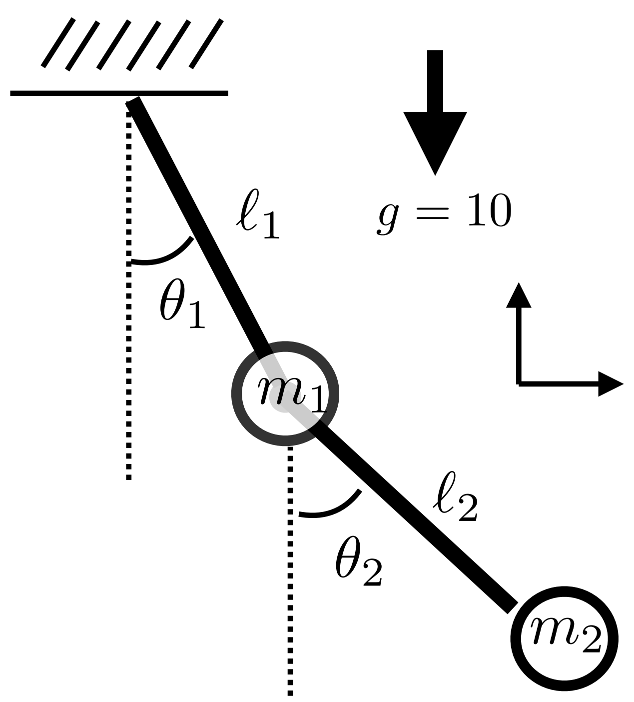
5 $$0 = \frac{1}{\sin\theta_2} m_2 \ddot{x}_2 + \frac{1}{\cos\theta_2}\left(m_2 g + m_2 \ddot{y}_2\right)$$

9 $$\boxed{0 = m_2 \ddot{\theta}_1 \ell_1 \cos(\theta_2 - \theta_1) + m_2 \dot{\theta}_1^2 \ell_1 \sin(\theta_2 - \theta_1) + m_2 \ddot{\theta}_2 \ell_2}$$

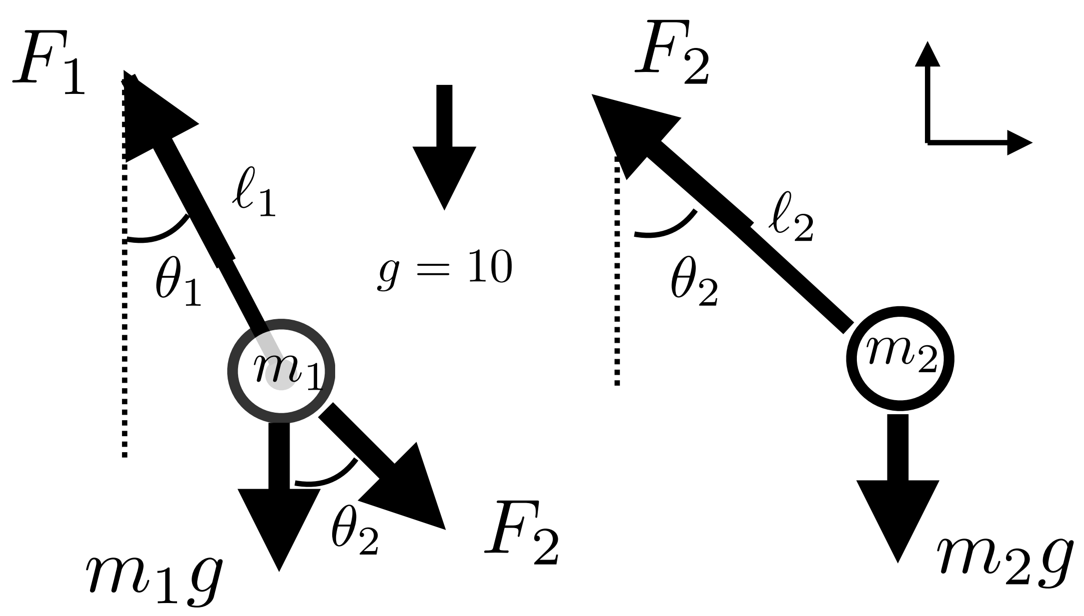
8 $$0 = \frac{1}{\sin\theta_1}\left(m_1 \ddot{x}_1 + m_2 \ddot{x}_2\right) + \frac{1}{\cos\theta_1}\left(m_1 g + m_2 g + m_1 \ddot{y}_1 + m_2 \ddot{y}_2\right)$$

10 $$\boxed{0 = (m_1 + m_2)\ddot{\theta}_1 \ell_1 + m_2 \ddot{\theta}_2 \ell_2 \cos(\theta_2 - \theta_1) - m_2 \dot{\theta}_2^2 \ell_2 \sin(\theta_2 - \theta_1) + \sin\theta_1 (m_1 + m_2) g}$$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$
$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$
$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$
$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$
$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Pendulum 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$
$$\dot{x}_2 = \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2$$
$$\ddot{x}_2 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1 + \ddot{\theta}_2 \ell_2 \cos\theta_2 - \dot{\theta}_2^2 \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$
$$\dot{y}_2 = \dot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_2 \ell_2 \sin\theta_2$$
$$\ddot{y}_2 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1 + \ddot{\theta}_2 \ell_2 \sin\theta_2 + \dot{\theta}_2^2 \ell_2 \cos\theta_2$$

Dynamics:

	$\sum (\text{forces})_x = m\ddot{x}$	$\sum (\text{forces})_y = m\ddot{y}$
Pendulum 1:	1 $-F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1$	2 $F_1 \cos\theta_1 - F_2 \cos\theta_2 - m_1 g = m_1 \ddot{y}_1$
Pendulum 2:	3 $-F_2 \sin\theta_2 = m_2 \ddot{x}_2$	4 $F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$

Writing in terms of pendulum angles… and simplifying…

9 $$\boxed{0 = m_2 \ddot{\theta}_1 \ell_1 \cos(\theta_2 - \theta_1) + m_2 \dot{\theta}_1^2 \ell_1 \sin(\theta_2 - \theta_1) + m_2 \ddot{\theta}_2 \ell_2}$$

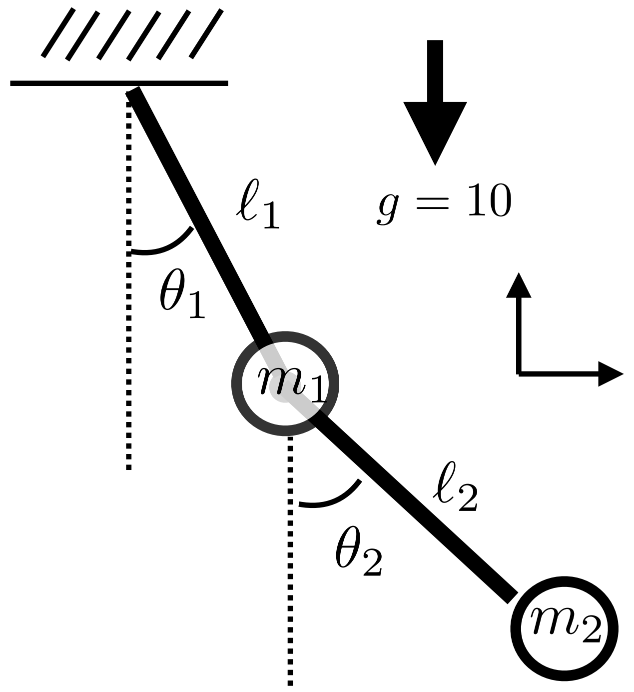
10 $$\boxed{0 = (m_1 + m_2)\ddot{\theta}_1 \ell_1 + m_2 \ddot{\theta}_2 \ell_2 \cos(\theta_2 - \theta_1) - m_2 \dot{\theta}_2^2 \ell_2 \sin(\theta_2 - \theta_1) + \sin\theta_1 (m_1 + m_2) g}$$

Dynamics…

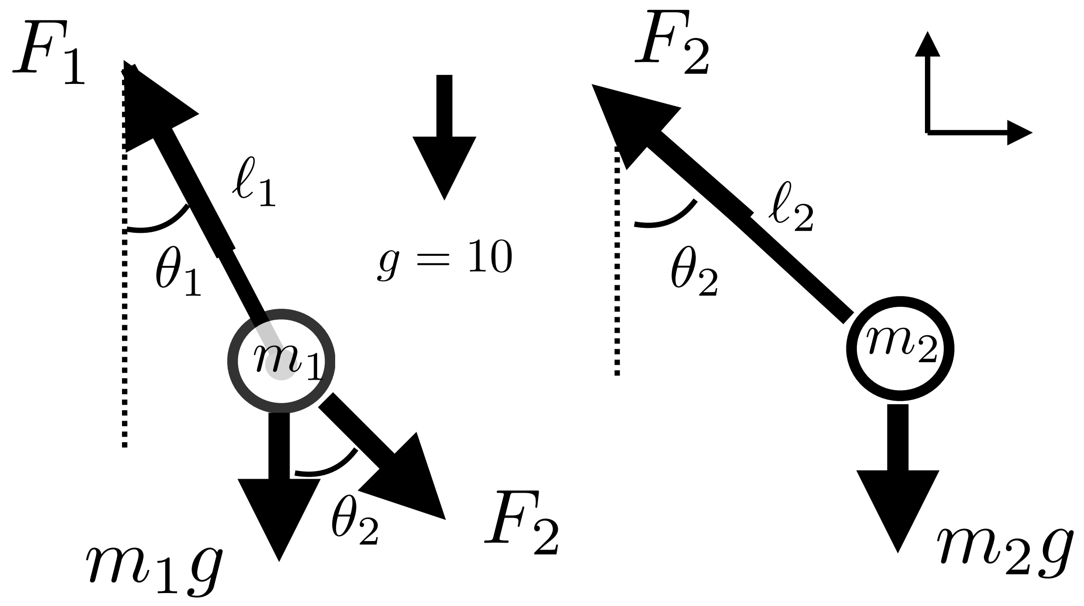
$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$
$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$
$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$
$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$
$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Pendulum 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$
$$\dot{x}_2 = \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2$$
$$\ddot{x}_2 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1 + \ddot{\theta}_2 \ell_2 \cos\theta_2 - \dot{\theta}_2^2 \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$
$$\dot{y}_2 = \dot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_2 \ell_2 \sin\theta_2$$
$$\ddot{y}_2 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1 + \ddot{\theta}_2 \ell_2 \sin\theta_2 + \dot{\theta}_2^2 \ell_2 \cos\theta_2$$

Dynamics:

$$\sum (\text{forces})_x = m\ddot{x} \qquad \sum (\text{forces})_y = m\ddot{y}$$

Pendulum 1:

1 $\quad -F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1$

2 $\quad F_1 \cos\theta_1 - F_2 \cos\theta_2 - m_1 g = m_1 \ddot{y}_1$

Pendulum 2:

3 $\quad -F_2 \sin\theta_2 = m_2 \ddot{x}_2$

4 $\quad F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$

Dynamics…

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

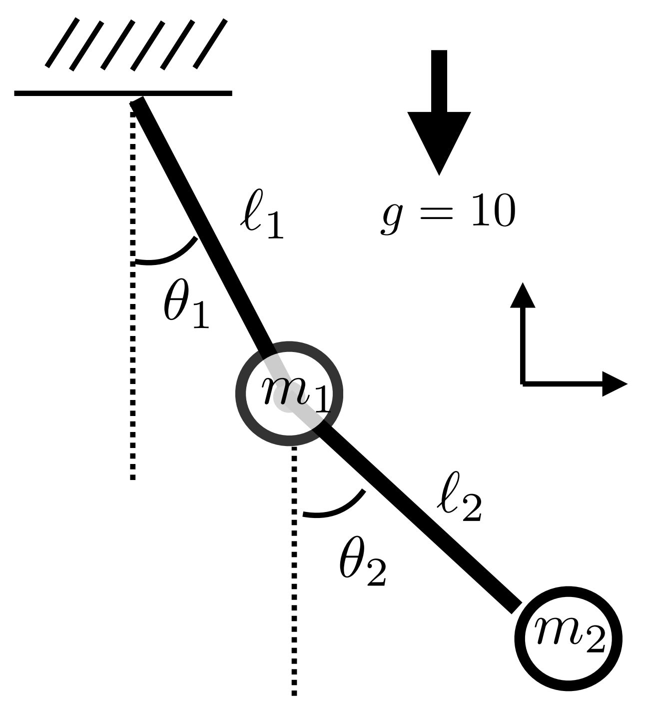
For linearization…

$$z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

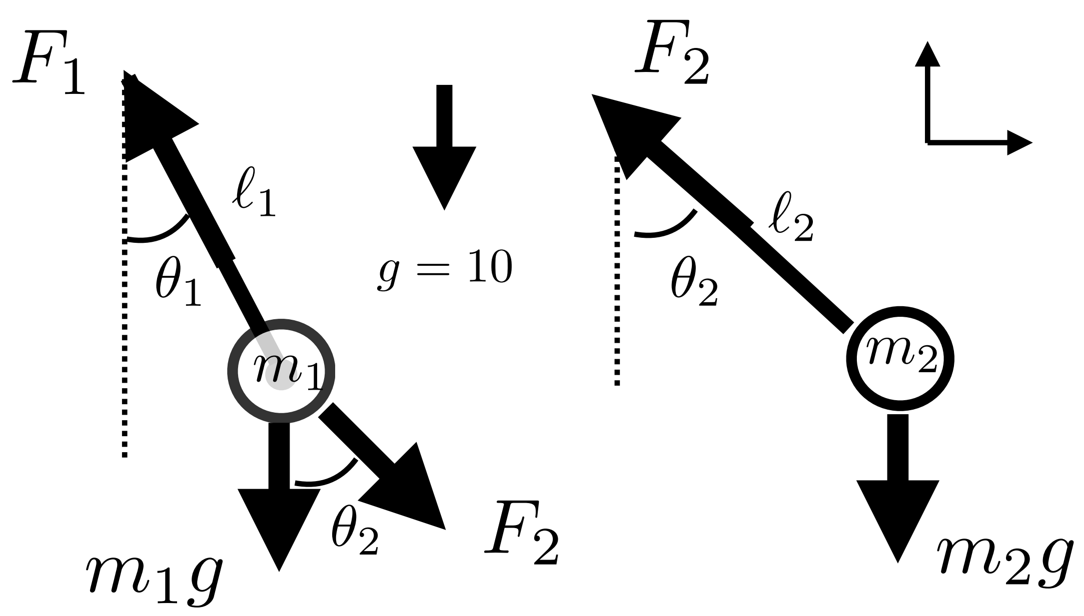
$$\frac{\partial \mathbf{a}}{\partial z} = \left[\frac{\partial \mathbf{M}^{-1}}{\partial \theta_1}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1}\mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2}\mathbf{F} \right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\left[\mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\frac{\partial \mathbf{M}}{\partial \dot{\theta}_1}}}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{M}^{-1}\overset{0}{\cancel{\frac{\partial \mathbf{M}}{\partial \dot{\theta}_2}}}\mathbf{M}^{-1}\mathbf{F} \right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$
$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$
$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$
$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$
$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Pendulum 2

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$
$$\dot{x}_2 = \dot{\theta}_1 \ell_1 \cos\theta_1 + \dot{\theta}_2 \ell_2 \cos\theta_2$$
$$\ddot{x}_2 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1 + \ddot{\theta}_2 \ell_2 \cos\theta_2 - \dot{\theta}_2^2 \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$
$$\dot{y}_2 = \dot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_2 \ell_2 \sin\theta_2$$
$$\ddot{y}_2 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1 + \ddot{\theta}_2 \ell_2 \sin\theta_2 + \dot{\theta}_2^2 \ell_2 \cos\theta_2$$

Dynamics:

$$\sum (\text{forces})_x = m\ddot{x} \qquad \sum (\text{forces})_y = m\ddot{y}$$

Pendulum 1:

1. $-F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1$
2. $F_1 \cos\theta_1 - F_2 \cos\theta_2 \; - m_1 g = m_1 \ddot{y}_1$

Pendulum 2:

3. $-F_2 \sin\theta_2 = m_2 \ddot{x}_2$
4. $F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$

Dynamics...

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

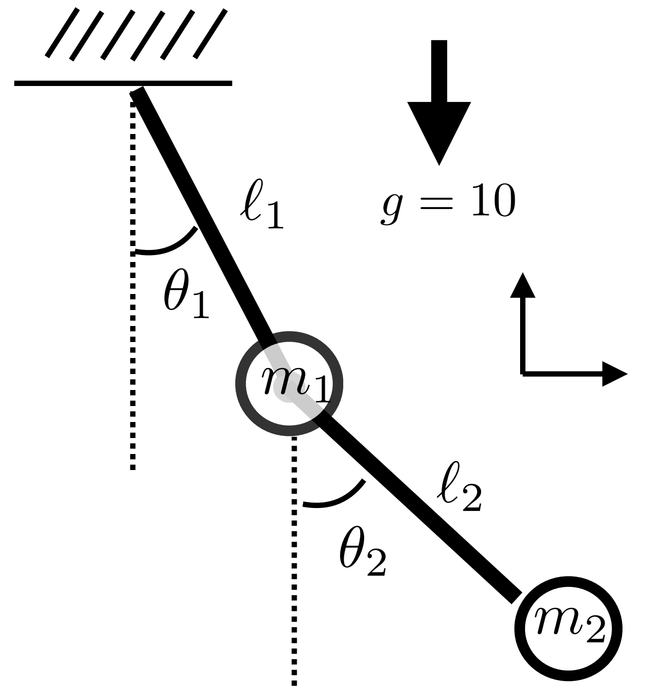
$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization...

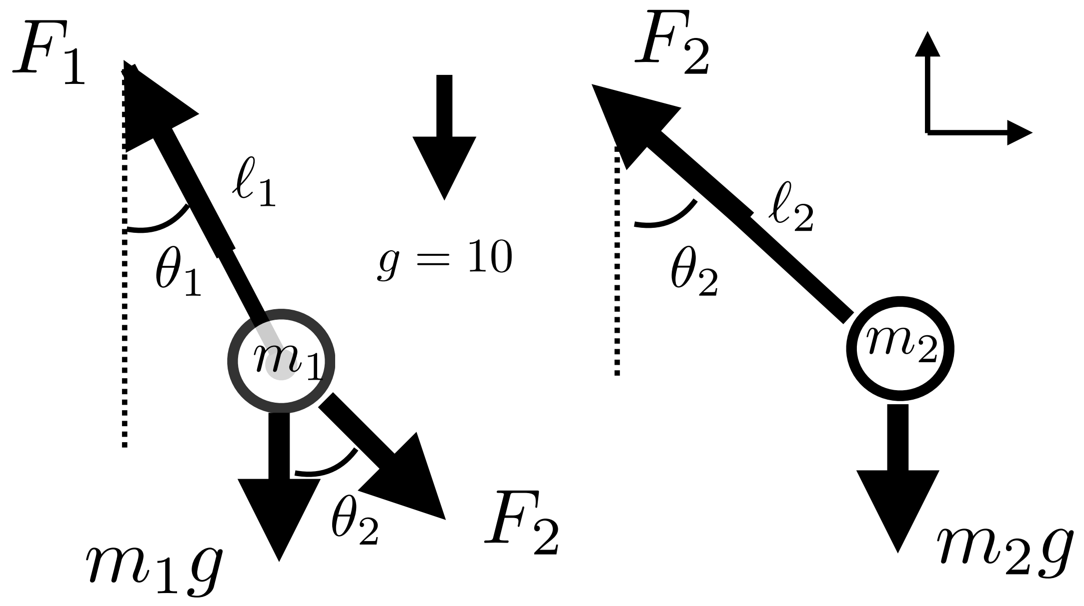
$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \frac{\partial \mathbf{M}^{-1}}{\partial \theta_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2}\mathbf{F} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} & \frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

Example: Double Pendulum



Free Body Diagrams



Kinematics

Pendulum 1

$$x_1 = \ell_1 \sin\theta_1$$

$$\dot{x}_1 = \dot{\theta}_1 \ell_1 \cos\theta_1$$

$$\ddot{x}_1 = \ddot{\theta}_1 \ell_1 \cos\theta_1 - \dot{\theta}_1^2 \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$

$$\dot{y}_1 = \dot{\theta}_1 \ell_1 \sin\theta_1$$

$$\ddot{y}_1 = \ddot{\theta}_1 \ell_1 \sin\theta_1 + \dot{\theta}_1^2 \ell_1 \cos\theta_1$$

Dynamics:

$$\sum (\text{forces})_x = m\ddot{x} \qquad \sum (\text{forces})_y = m\ddot{y}$$

Pendulum 1:

(1) $-F_1 \sin\theta_1 + F_2 \sin\theta_2 = m_1 \ddot{x}_1$

(2) $F_1 \cos\theta_1 - F_2 \cos\theta_2 - m_1 g = m_1 \ddot{y}_1$

Pendulum 2:

(3) $-F_2 \sin\theta_2 = m_2 \ddot{x}_2$

(4) $F_2 \cos\theta_2 - m_2 g = m_2 \ddot{y}_2$

Linearization

$$\dot{z} = f(z)$$

$$z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - & \frac{\partial \mathbf{a}}{\partial z} & - \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1} \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} & \frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \end{bmatrix}$$

Dynamics...

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1+m_2)\ell_1^2 & m_2\ell_2\ell_1\cos(\theta_2-\theta_1) \\ m_2\ell_2\ell_1\cos(\theta_2-\theta_1) & m_2\ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2\dot{\theta}_2^2\ell_2\ell_1\sin(\theta_2-\theta_1) - (m_1+m_2)\ell_1\sin\theta_1 g \\ -m_2\dot{\theta}_1^2\ell_1\ell_2\sin(\theta_2-\theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization...

$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \frac{\partial \mathbf{M}^{-1}}{\partial \theta_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2}\mathbf{F} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} & \frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

$$z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Example: Double Pendulum

Free Body Diagrams

$$\frac{\partial \mathbf{F}}{\partial z} = \begin{bmatrix} -m_2\dot{\theta}_2^2\ell_2\ell_1\cos(\theta_2-\theta_1) - (m_1+m_2)\ell_1\cos\theta_1 g & m_2\dot{\theta}_2^2\ell_2\ell_1\cos(\theta_2-\theta_1) & 0 & m_2 2\dot{\theta}_2\ell_2\ell_1\sin(\theta_2-\theta_1) \\ m_2\dot{\theta}_1^2\ell_1\ell_2\cos(\theta_2-\theta_1) & -m_2\dot{\theta}_1^2\ell_1\ell_2\cos(\theta_2-\theta_1) & -m_2 2\dot{\theta}_1\ell_1\ell_2\sin(\theta_2-\theta_1) & 0 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\begin{bmatrix} m_2\ell_2^2 & -m_2\ell_2\ell_1\cos(\theta_2-\theta_1) \\ -m_2\ell_2\ell_1\cos(\theta_2-\theta_1) & (m_1+m_2)\ell_1^2 \end{bmatrix}$$

$$\det(\mathbf{M}) = m_2\ell_1^2\ell_2^2\left(m_1 - m_2\sin^2(\theta_2-\theta_1)\right)$$

$$\frac{\partial \mathbf{M}}{\partial \theta_1} = \begin{bmatrix} 0 & m_2\ell_2\ell_1\sin(\theta_2-\theta_1) \\ m_2\ell_2\ell_1\sin(\theta_2-\theta_1) & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{M}}{\partial \theta_2} = \begin{bmatrix} 0 & -m_2\ell_2\ell_1\sin(\theta_2-\theta_1) \\ -m_2\ell_2\ell_1\sin(\theta_2-\theta_1) & 0 \end{bmatrix}$$

$m_1\ddot{y}_1$

Linearization

$$\dot{z} = f(z)$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - & \frac{\partial \mathbf{a}}{\partial z} & - \end{bmatrix} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1}\left[\frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{0} \quad \mathbf{0}\right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \end{bmatrix}$$

Dynamics...

$$\mathbf{a} = \mathbf{M}^{-1}\mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1+m_2)\ell_1^2 & m_2\ell_2\ell_1\cos(\theta_2-\theta_1) \\ m_2\ell_2\ell_1\cos(\theta_2-\theta_1) & m_2\ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2\dot{\theta}_2^2\ell_2\ell_1\sin(\theta_2-\theta_1) - (m_1+m_2)\ell_1\sin\theta_1 g \\ -m_2\dot{\theta}_1^2\ell_1\ell_2\sin(\theta_2-\theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization...

$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \frac{\partial \mathbf{M}^{-1}}{\partial \theta_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1}\mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2}\mathbf{F} \end{bmatrix} + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1}\left[\frac{\partial \mathbf{M}}{\partial \theta_1}\mathbf{M}^{-1}\mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2}\mathbf{M}^{-1}\mathbf{F} \quad \mathbf{0} \quad \mathbf{0}\right] + \mathbf{M}^{-1}\frac{\partial \mathbf{F}}{\partial z}$$

Linearization Stability - Case 1

$m_1 = m_2 = m_3 = 1$

$\theta_1 = \theta_2 = 0$ $\ell_1 = \ell_2 = 1$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -20. & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

$\lambda_{1,2} = 0 \qquad \lambda_{3,4} = \pm 4.472i$

Comments

Since there is no damping the system is only marginally stable (real parts of evals = 0) and will oscillate forever.

Free Body Diagrams

$$\ldots \quad 0 \qquad m_2 2\dot{\theta}_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1)$$
$$\ldots \quad -m_2 2\dot{\theta}_1 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \qquad 0$$

$$\ldots = \begin{bmatrix} 0 & m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$\ldots = \begin{bmatrix} 0 & -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$m_1 \ddot{y}_1$

Linearization

$$\dot{z} = f(z) \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - \; \frac{\partial \mathbf{a}}{\partial z} & - \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \end{bmatrix}$$

Dynamics...

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization...

$$\frac{\partial \mathbf{a}}{\partial z} = \left[\frac{\partial \mathbf{M}^{-1}}{\partial \theta_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2} \mathbf{F} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z}$$

Linearization Stability - Case 2

$$m_1 = m_2 = m_3 = 1$$

$$\theta_1 = \theta_2 = 0 \qquad \ell_1 = \ell_2 = 100$$

… other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

$$\lambda_{1,2} = 0 \qquad \lambda_{3,4} = \pm 0.4472i$$

Comments

Since there is no damping the system is only marginally stable (real parts of evals = 0) and will oscillate forever.
Since the pendulums are longer, the oscillation rate is slower (the imaginary parts of eigenvalues are smaller)

Free Body Diagrams

$$\ldots) \quad 0 \quad m_2 2\dot{\theta}_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \Big]$$
$$\ldots_1) \quad -m_2 2\dot{\theta}_1 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \quad 0$$

$$\theta_2$$

$$= \begin{bmatrix} 0 & m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$m_1 \ddot{y}_1$$

Linearization

$$\dot{z} = f(z)$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - & \frac{\partial \mathbf{a}}{\partial z} & - \end{bmatrix} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} & \end{bmatrix}$$

Dynamics…

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization…

$$\frac{\partial \mathbf{a}}{\partial z} = \left[\frac{\partial \mathbf{M}^{-1}}{\partial \theta_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2} \mathbf{F} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z}$$

Linearization Stability - Case 2

$m_1 = m_2 = m_3 = 1$ $\theta_1 = \theta_2 = \pi$ $\ell_1 = \ell_2 = 1$

... other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 20. & 0 & 0 & 0 \\ -20. & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

$\lambda_{1,2} = 0$ $\lambda_{3,4} = \pm 4.472$

Comments

Since the pendulums are vertical the system is unstable with a positive real eigenvalue.

Free Body Diagrams

$$\cdots) \quad 0 \quad m_2 2\dot{\theta}_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1)]$$
$$\cdots_1) \quad -m_2 2\dot{\theta}_1 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \quad 0$$

θ_2

$$= \begin{bmatrix} 0 & m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$m_1 \ddot{y}_1$

Linearization

$$\dot{z} = f(z)$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - & \frac{\partial \mathbf{a}}{\partial z} & - \end{bmatrix} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \end{bmatrix}$$

Dynamics...

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin \theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization...

$$\frac{\partial \mathbf{a}}{\partial z} = \left[\frac{\partial \mathbf{M}^{-1}}{\partial \theta_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1} \mathbf{F} \quad \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2} \mathbf{F} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \left[\frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} \quad \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} \quad \mathbf{0} \quad \mathbf{0} \right] + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z}$$

Linearization Stability - Case 2

$m_1 = m_2 = m_3 = 1$ $\qquad \theta_1 = \theta_2 = \pi$ $\qquad \ell_1 = \ell_2 = 100$

… other states/forces = 0

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues

$$\lambda_{1,2} = 0 \qquad \lambda_{3,4} = \pm 0.447$$

Comments

Since the pendulums are vertical the system is unstable with a positive real eigenvalue. Since the pendulums are longer it takes longer for them to fall over and the eigenvalues have smaller magnitudes.

Free Body Diagrams

$$\begin{bmatrix} 0 & m_2 2\dot{\theta}_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ -m_2 2\dot{\theta}_1 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) \\ -m_2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) & 0 \end{bmatrix}$$

$m_1 \ddot{y}_1$

Linearization

$$\dot{z} = f(z) \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \mathbf{0} & I \\ - & \frac{\partial \mathbf{a}}{\partial z} \ - \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & I \\ -\mathbf{M}^{-1} \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} & \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \end{bmatrix}$$

Dynamics…

$$\mathbf{a} = \mathbf{M}^{-1} \mathbf{F}$$

$$\underbrace{\begin{bmatrix} (m_1 + m_2)\ell_1^2 & m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) \\ m_2 \ell_2 \ell_1 \cos(\theta_2 - \theta_1) & m_2 \ell_2^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} m_2 \dot{\theta}_2^2 \ell_2 \ell_1 \sin(\theta_2 - \theta_1) - (m_1 + m_2)\ell_1 \sin\theta_1 g \\ -m_2 \dot{\theta}_1^2 \ell_1 \ell_2 \sin(\theta_2 - \theta_1) \end{bmatrix}}_{\mathbf{F}}$$

Linearization…

$$\frac{\partial \mathbf{a}}{\partial z} = \begin{bmatrix} \frac{\partial \mathbf{M}^{-1}}{\partial \theta_1} \mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \theta_2} \mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_1} \mathbf{F} & \frac{\partial \mathbf{M}^{-1}}{\partial \dot{\theta}_2} \mathbf{F} \end{bmatrix} + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z} \qquad z = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial z} = -\mathbf{M}^{-1} \begin{bmatrix} \frac{\partial \mathbf{M}}{\partial \theta_1} \mathbf{M}^{-1} \mathbf{F} & \frac{\partial \mathbf{M}}{\partial \theta_2} \mathbf{M}^{-1} \mathbf{F} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathbf{M}^{-1} \frac{\partial \mathbf{F}}{\partial z}$$