Kinematics, Dynamics, & Linearization

Dynamics & Modeling

Major sources: Kaare Brandt Petersen
Michael Syskind Pedersen

Major references: The Matrix Cookbook - Petersen, Pedersen

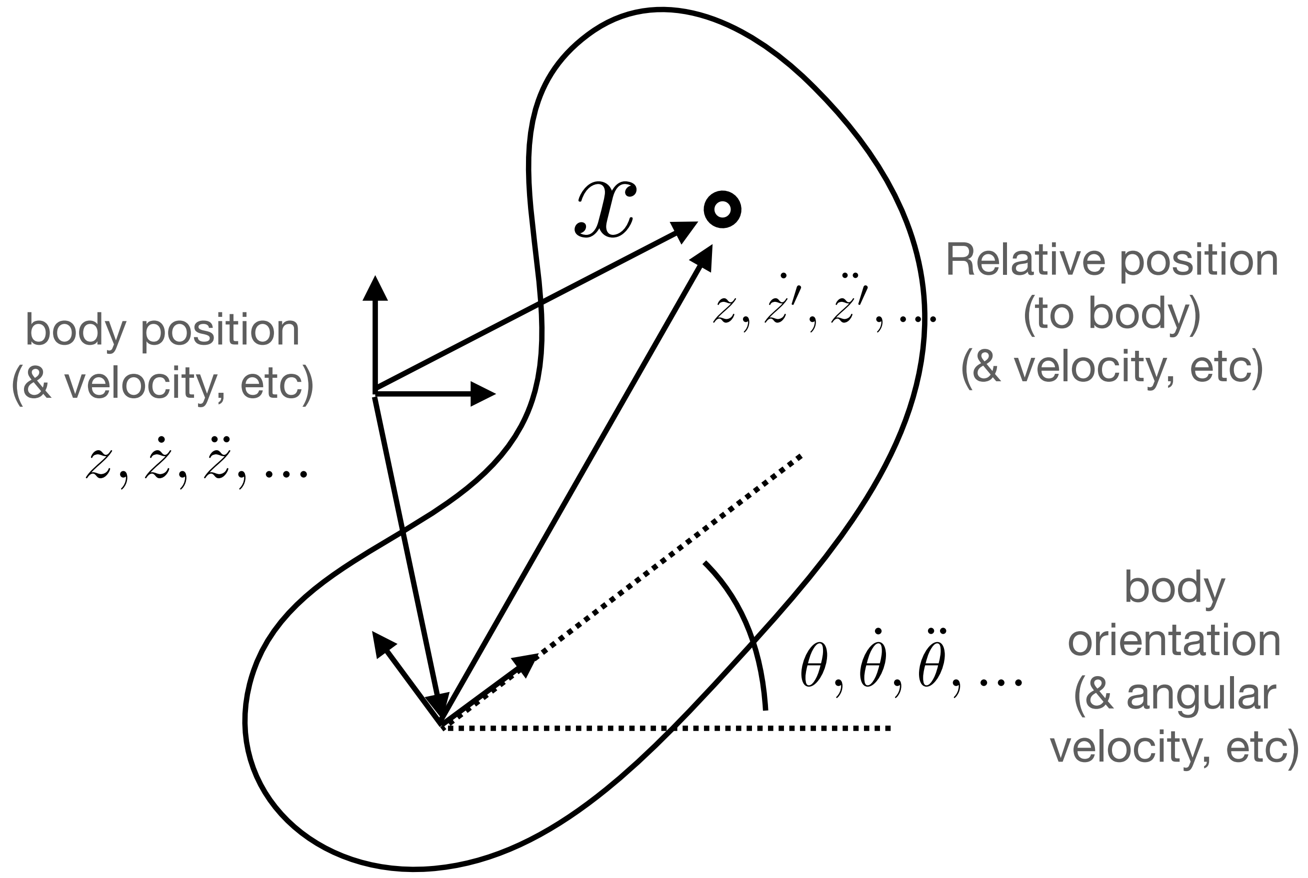
Winter 2022 - Dan Calderone

Kinematics 2D

useful for most homework problems

… for mechanical systems,
higher order motion can be quite complicated

2D:



Rotation Matrix Derivation:

Note:
rotation "axis"
- out of the page

$$\hat{\omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R(\theta) = e^{\hat{\omega}\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\dot{R}(\theta) = \dot{\theta}\hat{\omega}e^{\hat{\omega}\theta} = \dot{\theta}\hat{\omega}R(\theta)$$

$$\ddot{R}(\theta) = \ddot{\theta}\hat{\omega}e^{\hat{\omega}\theta} + \dot{\theta}^2\hat{\omega}^2 e^{\hat{\omega}\theta}$$

$$= \ddot{\theta}\hat{\omega}R(\theta) + \dot{\theta}^2\hat{\omega}^2 R(\theta)$$

Velocity Terms:

$$\dot{x} = \underset{\substack{\downarrow \\ \text{Body velocity}}}{\dot{z}} + \underset{\substack{\downarrow \\ \text{Angular velocity}}}{\dot{\theta}\hat{\omega}R(\theta)z'} + \underset{\substack{\downarrow \\ \text{Relative velocity}}}{R(\theta)\dot{z}'}$$

Position: $x = z + R(\theta)z'$

Velocity:

$$\dot{x} = \dot{z} + \dot{R}(\theta)z' + R(\theta)\dot{z}'$$

$$= \dot{z} + \dot{\theta}\hat{\omega}R(\theta)z' + R(\theta)\dot{z}'$$

Acceleration Terms:

$$\ddot{x} = \underset{\substack{\downarrow \\ \text{Body acceleration}}}{\ddot{z}} + \underset{\substack{\downarrow \\ \text{Angular acceleration}}}{\ddot{\theta}\hat{\omega}R(\theta)z'} + \underset{\substack{\downarrow \\ \text{Centripetal acceleration}}}{\dot{\theta}^2\hat{\omega}^2R(\theta)z'} + \underset{\substack{\downarrow \\ \text{Coriolis acceleration}}}{2\dot{\theta}\hat{\omega}R(\theta)\dot{z}'} + \underset{\substack{\downarrow \\ \text{Relative acceleration}}}{R(\theta)\ddot{z}'}$$

Acceleration:

$$\ddot{x} = \ddot{z} + \ddot{R}(\theta)z' + 2\dot{R}(\theta)\dot{z}' + R(\theta)\ddot{z}'$$

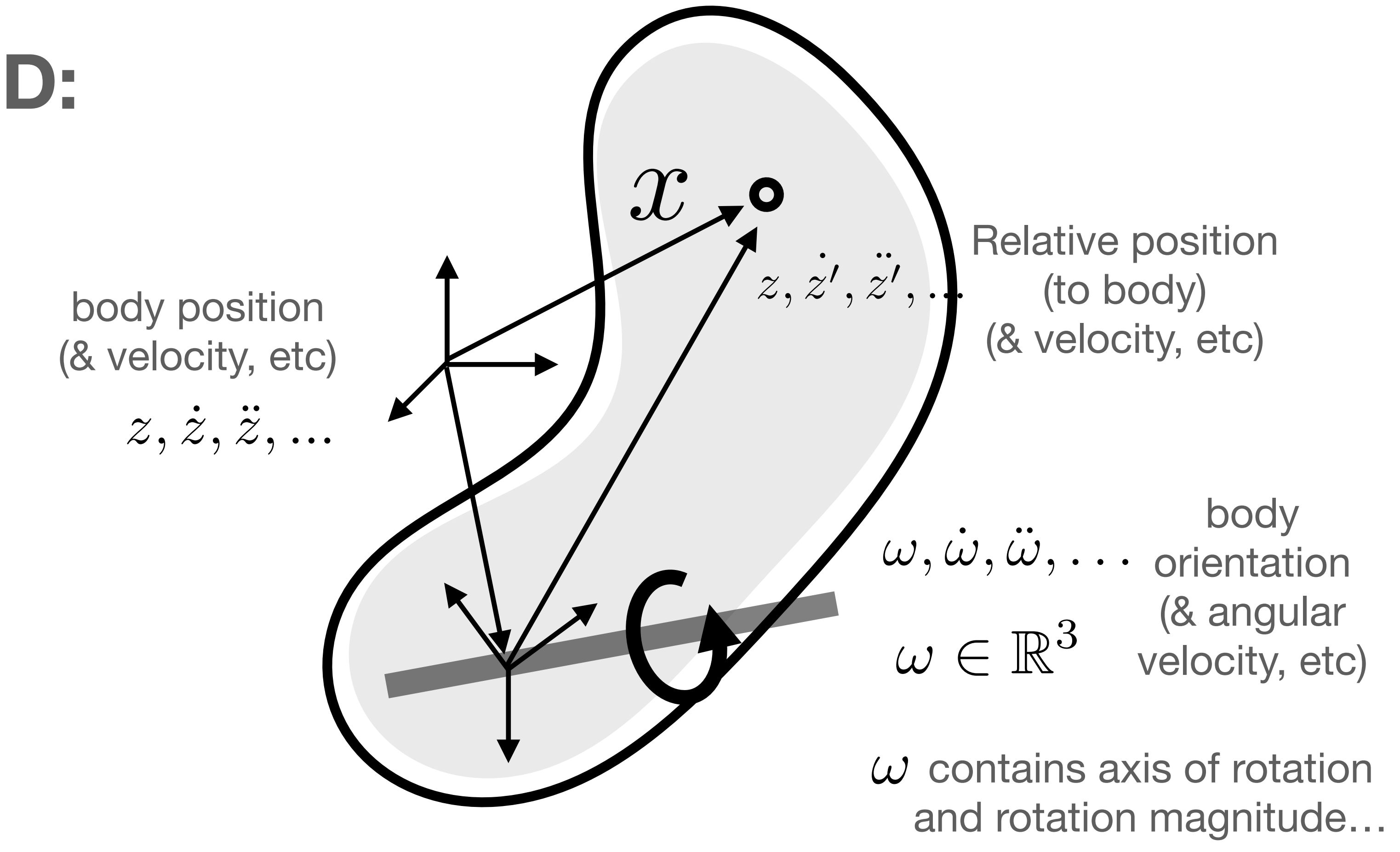
$$= \ddot{z} + \ddot{\theta}\hat{\omega}R(\theta)z' + \dot{\theta}^2\hat{\omega}^2R(\theta)z' + 2\dot{\theta}\hat{\omega}R(\theta)\dot{z}' + R(\theta)\ddot{z}'$$

Kinematics 3D

generalization

… for mechanical systems,
higher order motion can be quite complicated

3D:



…for fixed axis rotations with axis ξ: $\omega = \theta\xi,\ \dot\omega = \dot\theta\xi,\ \ddot\omega = \ddot\theta\xi$

Rotation Matrix Derivation:

$$\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \qquad \hat\omega = \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$$

$$R(\omega) = e^{\hat\omega}$$

$$\dot R(\omega) = \dot{\hat\omega} e^{\hat\omega}$$

$$\ddot R(\omega) = \ddot{\hat\omega} e^{\hat\omega} + \dot{\hat\omega}^2 e^{\hat\omega}$$

Velocity Terms:

$$\dot x = \underbrace{\dot z}_{\text{Body velocity}} + \underbrace{\dot{\hat\omega} R(\omega) z'}_{\text{Angular velocity}} + \underbrace{R(\omega)\dot z'}_{\text{Relative velocity}}$$

Position: $x = z + R(\omega)z'$

Velocity:

$$\dot x = \dot z + \dot R(\omega) z' + R(\omega)\dot z'$$
$$= \dot z + \dot{\hat\omega} R(\omega) z' + R(\omega)\dot z'$$

Acceleration Terms:

$$\ddot x = \underbrace{\ddot z}_{\text{Body acceleration}} + \underbrace{\ddot{\hat\omega} R(\omega) z'}_{\text{Angular acceleration}} + \underbrace{\dot{\hat\omega}^2 R(\omega) z'}_{\text{Centripetal acceleration}} + \underbrace{2\dot{\hat\omega} R(\omega)\dot z'}_{\text{Coriolis acceleration}} + \underbrace{R(\omega)\ddot z'}_{\text{Relative acceleration}}$$

Acceleration:

$$\ddot x = \ddot z + \ddot R(\omega) z' + 2\dot R(\omega)\dot z' + R(\omega)\ddot z'$$
$$= \ddot z + \ddot{\hat\omega} R(\omega) z' + \dot{\hat\omega}^2 R(\omega) z' + 2\dot{\hat\omega} R(\omega)\dot z' + R(\omega)\ddot z'$$